

Solution for exercise 10, ex. 1 final examination 16/12-1991

Question 1 The model is a discrete time Markov chain. A possible definition of states could be

0: The programme has stopped.

1-4: The programme is operating safely in level i .

5-8: The programme is operating in level $i-4$, the critical error is not detected.

The transition matrix \mathbf{A} is

$$\mathbf{A} = \begin{bmatrix} 1 & \vec{0} & \vec{0} \\ \vec{0} & \mathbf{P} & \mathbf{0} \\ \vec{r} & \text{Diag}(\mathbf{q}_i)\mathbf{P} & \text{Diag}(\mathbf{S}_i)\mathbf{P} \end{bmatrix}$$

Hvor $\mathbf{P} = \{p_{ij}\}$

$$\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad \text{Diag}(\mathbf{S}_i) = \begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & S_4 \end{bmatrix} \quad S_i = 1 - r_i - q_i$$

$$\text{Diag}(\mathbf{q}_i) = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}$$

Eller uden matrix notation:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & p_{13} & p_{14} & 0 & 0 & 0 & 0 \\ 0 & p_{21} & p_{22} & p_{23} & p_{24} & 0 & 0 & 0 & 0 \\ 0 & p_{31} & p_{32} & p_{33} & p_{34} & 0 & 0 & 0 & 0 \\ 0 & p_{41} & p_{42} & p_{43} & p_{44} & 0 & 0 & 0 & 0 \\ r_1 & q_1 p_{11} & q_1 p_{12} & q_1 p_{13} & q_1 p_{14} & S_1 p_{11} & S_1 p_{12} & S_1 p_{13} & S_1 p_{14} \\ r_2 & q_2 p_{21} & q_2 p_{22} & q_2 p_{23} & q_2 p_{24} & S_2 p_{21} & S_2 p_{22} & S_2 p_{23} & S_2 p_{24} \\ r_3 & q_3 p_{31} & q_3 p_{32} & q_3 p_{33} & q_3 p_{34} & S_3 p_{31} & S_3 p_{32} & S_3 p_{33} & S_3 p_{34} \\ r_4 & q_4 p_{41} & q_4 p_{42} & q_4 p_{43} & q_4 p_{44} & S_4 p_{41} & S_4 p_{42} & S_4 p_{43} & S_4 p_{44} \end{bmatrix}$$

Question 2 With reasonable assumptions on \mathbf{P} (i.e. irreducible) we get

| | | |
|----------|---|--------------------|
| Tilstand | 0 | Absorbing |
| | 1 | Positive recurrent |
| | 2 | Positive recurrent |
| | 3 | Positive recurrent |
| | 4 | Positive recurrent |
| | 5 | Transient |
| | 6 | Transient |
| | 7 | Transient |
| | 8 | Transient |

Question 3 The system becomes stable by reaching on of the states 1-4. The process is ergodic from then on. The process is reversible ergodic Markov chain in discrete time.

Question 4 We obtain the following steady state equations

$$\pi_i = 3^{1-i} \pi_1$$

$$\sum_{i=1}^4 3^{i-1} \pi_1 = 1 \Leftrightarrow 40\pi_1 = 1$$

$$\pi_1 = \frac{1}{40}$$

The sum $\sum_{i=1}^4 3^{i-1}$ can be obtained by using $\sum_{i=1}^4 3^{i-1} = \frac{1-3^4}{1-3} = 40$.