IMM - DTU

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BFN/bfn

Question 1 The two models can be found in the MAP-note and we find: First the Erlang renewal process:

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

and then the process with hyperexponentially distributed times

$$\mathbf{C}_0 = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix} \qquad \mathbf{C}_1 = \begin{bmatrix} \mu_1 p & \mu_1(1-p) \\ \mu_2 p & \mu_2(1-p) \end{bmatrix}$$

Question 2 The combined stream is a superposition of the two primary streams and is thus found as their superposition i.e. a MAP with $(\mathbf{D}_0 \oplus \mathbf{C}_0, \mathbf{D}_1 \oplus \mathbf{C}_1)$.

Question 3 We first notice that $\vec{p}_1 \otimes \vec{p}_2$ is a probability vector, i.e. $(\vec{p}_1 \otimes \vec{p}_2)(\vec{e} \otimes \vec{e}) = (\vec{p}_1 \otimes \vec{e})(\vec{p}_2 \otimes \vec{e}) = 1 \cdot 1 = 1$. And then show that the asserted vector satisfies the balance equations.

$$(\vec{p}_1 \otimes \vec{p}_2)(\mathbf{A}_1 \oplus \mathbf{A}_2) = \vec{0}$$

The proof is as follows

$$(\vec{p}_1 \otimes \vec{p}_2)(\mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_2) = (\vec{p}_1 \otimes \vec{p}_2)(\mathbf{A}_1 \otimes \mathbf{I}) + (\vec{p}_1 \otimes \vec{p}_2)(\mathbf{I} \otimes \mathbf{A}_2)$$

we now use the hint and find

$$(\vec{p}_1\mathbf{A}_1) \otimes (\vec{p}_2\mathbf{I}) + (\vec{p}_1\mathbf{I}) \otimes (\vec{p}_2\mathbf{A}_2) = \vec{0} \otimes \vec{p}_2 + \vec{p}_1 \otimes \vec{0} = \vec{0}$$

Question 4 The imbedded Markov chain, for the MAP we consider, emerges from $\mathbf{D} \oplus \mathbf{C}$ and we can thus use the result from question 3. The problem has thus been reduced to finding the the stationary state vectors for each of the contributing MAPs. These can be found either in the note material or directly by calculation and we find:

$$\vec{\theta}_D = \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \vec{\theta}_C = \left(\frac{\frac{p}{\mu_1}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}, \frac{\frac{1-p}{\mu_2}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}\right)$$

The stationary probability vector is thus:

$$\vec{\theta} = \vec{\theta}_D \vec{\theta}_C = \left(\frac{1}{2}\frac{\frac{p}{\mu_1}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}, \frac{1}{2}\frac{\frac{1-p}{\mu_2}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}, \frac{1}{2}\frac{\frac{p}{\mu_1}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}, \frac{1}{2}\frac{\frac{1-p}{\mu_2}}{\frac{p}{\mu_1} + \frac{1-p}{\mu_2}}\right)$$

Question 5 The fundamental rate can be found by

$$\lambda^\star = (\vec{\theta}_D \otimes \vec{\theta}_C)(\mathbf{D}_1 \oplus \mathbf{C}_1)(\vec{e} \otimes \vec{e}) = (\vec{\theta}_D \mathbf{D}_1 \otimes \vec{\theta}_C + \vec{\theta}_D \otimes \vec{\theta}_C \mathbf{C}_1)(\vec{e} \otimes \vec{e}) = \lambda^\star_D + \lambda^\star_C$$

Thus the fundamental rate is just the sum of the fundamental rates for each of the two processes. In this case the rate becomes

$$\lambda^\star = \frac{\lambda}{2} + \frac{\mu_1\mu_2}{\mu_2 p + \mu_1(1-p)}$$

Question 6 As the process is stationary the mean number is just the fundamental rate multiplied by the length of the time interval i.e. $\lambda^\star t$.

Question 7 We use the formulas (128) and (129) p.35 in Wolff. And find the mean number to be $104\lambda^\star t$ and the variance to be $\lambda^\star t 350^2 + \sigma_p^2(t)104^2$.