IMM - DTU

02407 Stochastic Processes
2008-9-23
BFN/bfn

Solution exercise 1

Question 1 An adequate model would be that successive pairs have behave independently. For each pair there will be a probability p_1 that the male will buy an ice-cream. The distribution in question will be geometric.

$$f(x) = (1-p_1)^{x-1}p_1$$

$$F(x) = 1-(1-p_1)^x$$

With the phase type representation

$$\mathbf{T} = [1-p_1], \qquad \vec{\alpha} = (1)$$

Question 2 We will apply a similar model for the time (number of pairs) which will pass before a female buys an ice cream with probability p_2. The distribution is now found as the distribution of the minimum of two phase type distributed random variables. The representation of this phase type distribution is

$$\mathbf{L} = [(1-p_1)(1-p_2)], \qquad \vec{\gamma} = (1)$$

Once again the distribution is geometric. The geometric distribution is the discrete parallel of the exponential distribuion. This result shows that the minimum of two (n) independent geometrically distributed random variables follows a geometric distribution.

Question 3 The distribution of the time until the first couple buys an ice cream was found in question 2. The time between having two couples bying an ice cream follows the same distribution and the time until two couples have bought an ice cream is then the sum of two phase type distributed variables. The representation is

$$\mathbf{L} = \begin{bmatrix} (1-p_1)(1-p_2) & 1-(1-p_1)(1-p_2) \\ 0 & (1-p_1)(1-p_2) \end{bmatrix}, \qquad \vec{\gamma} = (1,0)$$

The distribution is the distribution of a sum of two geometrically distributed random variables and is a negative binomial distribution. With $p = (1-p_1)(1-p_2)$ we get

$$\mathbf{L}^2 = \begin{bmatrix} p^2 & p\cdot(1-p)+(1-p)\cdot p \\ 0 & p^2 \end{bmatrix} = \begin{bmatrix} p^2 & 2p(1-p) \\ 0 & p^2 \end{bmatrix}$$

$$\mathbf{L}^3 = \begin{bmatrix} p^3 & p^2 \cdot (1-p) + 2p(1-p) \cdot p \\ 0 & p^3 \end{bmatrix} = \begin{bmatrix} p^3 & 3p^2(1-p) \\ 0 & p^3 \end{bmatrix}$$

And by induction

$$\mathbf{L}^n = \begin{bmatrix} p^n & \begin{pmatrix} n \\ 1 \end{pmatrix} p^{n-1}(1-p) \\ 0 & p^n \end{bmatrix}, \qquad n \geq 1$$

The probability density function is given by $f(x) = \vec{\gamma}\mathbf{L}^{x-1}\vec{L^0}, x \geq 1$:

$$\begin{aligned}
f(0) &= 0 \\
f(1) &= (1,0)\mathbf{L}^0 \begin{bmatrix} 0 \\ 1-p \end{bmatrix} = (1,0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1-p \end{bmatrix} = 0 \\
f(x) &= (1,0) \begin{bmatrix} p^{x-1} & \begin{pmatrix} x-1 \\ 1 \end{pmatrix} p^{x-2}(1-p) \\ 0 & p^{x-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1-p \end{bmatrix} \\
&= \begin{pmatrix} x-1 \\ 1 \end{pmatrix} p^{x-2}(1-p)^2, \qquad x \geq 2
\end{aligned}$$

The density is recognised as being the density of a negative binomial distribution.

Question 4 We consider the largest of two possible time events: The time until the first male and the first female buys an ice cream. Since both of these follows a phase type distribution the maximum will also follow a phase type distribution. That distribution has the representation given below

$$\mathbf{L} = \begin{bmatrix} (1-p_1)(1-p_2) & (1-p_1)p_2 & p_1(1-p_2) \\ 0 & 1-p_1 & 0 \\ 0 & 0 & 1-p_2 \end{bmatrix}, \qquad \vec{\gamma} = (1,0,0)$$