IMM - DTU

04141 Stochastic Processes
1997-12-5
BFN/bfn

Solution to exercise 20

Question 1

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -(\lambda+\mu) & \lambda \\ 0 & 0 & -(\lambda+2\mu) \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 2\mu & 0 \end{bmatrix}$$

Question 2

$$\mathbf{D} = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -(\lambda+2\mu) \end{bmatrix}$$

$$\vec{\theta} = \left(\frac{1}{1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2}, \frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2}, \frac{\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2}{1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2} \right) \qquad \vec{\lambda} = \mathbf{D}_1\vec{e} = \begin{bmatrix} 0 \\ \mu \\ 2\mu \end{bmatrix}$$

$$\lambda^\star = \vec{\theta}\vec{\lambda} = \lambda(\theta_0+\theta_1) = \lambda(1-\theta_2)$$

The average rate of completed services equals the rate of accepted arrivals. Note that the example is Erlangs blocking system.

Question 3 The MAP is a PH-renewal process in the case $c = 1$. The mean is $\frac{1}{\lambda}+\frac{1}{\mu}$. The variance is $\frac{1}{\lambda^2}+\frac{1}{\mu^2}$.

Question 4 We need the embedded Markov chain immediately after service completions. By applying (1.28) p.14 in the MAP note

$$(-\mathbf{D}_0)^{-1} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\frac{1}{2\mu} \\ 0 & \frac{1}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\frac{1}{2\mu} \\ 0 & 0 & \frac{1}{2\mu} \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The expression for $(-\mathbf{D}_0)^{-1}$ can be found by probabilistic considerations. In order to apply (1.30) p.15 we only need to calculate $\vec{\phi}$.

$$\vec{\phi} = \left(\frac{\mu}{\mu+\lambda}, \frac{\lambda}{\mu+\lambda}, 0 \right)$$

Question 5

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda \\ 0 & 0 & 3\mu & -(\lambda+3\mu) \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

The MAP is a PH-renewal process.

Question 6 Since the process is a renewal process the covariance is 0.

Question 7 Consider a birth and death process with the following structure

$$\mathbf{A} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -(\lambda_{m-1}+\mu_{m-1}) & \lambda_{m-1} \\ 0 & 0 & 0 & 0 & \dots & \mu_m & -\mu_m \end{bmatrix}$$

The MAP describing downward transitions in this process is

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -(\lambda_2+\mu_2) & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -(\lambda_{m-1}+\mu_{m-1}) & \lambda_{m-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\mu_m \end{bmatrix}$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \mu_m & 0 \end{bmatrix}$$

Question 8

$$\mathbf{D}_0 = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0,0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{1,1} & \mathbf{A}_{1,0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{2,1} & \mathbf{A}_{2,0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{3,1} & \mathbf{A}_{3,0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{4,1} & \mathbf{A}_{4,0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,1} \end{bmatrix}$$

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{2,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{3,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{4,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,2} & \mathbf{0} \end{bmatrix}$$