Solution to exercise 27

Question 1: The system can be modeled as a continuous time Markov chain with 3 states:

1. Idle
2. Busy with no order in the book
3. Busy with one order in the book

We find the generator

\[
G = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\]

where \( \lambda = 1 \) is the arrival rate of requests, and \( \mu = 4 \) is the rate of completing jobs.

First, we find the steady-state distributions as in general for a birth-death process:

\[
\pi_i = c \cdot \rho^i \quad \text{for } i = 1, 2, 3
\]

where \( \rho = \lambda/\mu \) and \( c = (\rho + \rho^2 + \rho^3)^{-1} \) is a normalization constant making the probabilities sum to one, \( \pi_1 + \pi_2 + \pi_3 = 1 \).

Alternatively, we could find the stationary distribution directly numerically e.g. using \texttt{matlab}:

```matlab
piv = null(G');
piv = piv/sum(piv);
piv(3)
```

This gives a numeric value of 0.0476 which is 1/21.

Next, we realize that the fraction of customers arriving when the system is in state 2 must equal the fraction of time spent in state 2, and thus the stationary probability of state 2, \( \pi_2 \). This is the PASTA property (Poisson arrivals see time averages). To see this, assume that the system is stationary. The probability of a new customer arriving in the time interval \((t, t + h]\) is independent of which state the system is in at time \(t\). So given that a customer arrives, the probability that the system is in state 2 is still \(\pi_2\).

So the fraction of customers rejected is \(\pi_2 = 1/21\) or approximately 0.05.

Question 2: The fraction of time where the equipment is in use is equal to the probability that the system at a given fixed time is in state 2 or 3, i.e. \(\pi_2 + \pi_3 = 5/21\) or approximately 0.24.

Question 3: The states are

0: Idle
1: In use, no order in book
2: Rented, no order in book
3: In use, order in book
4: Rented, order in book

The generator is

\[
G = \begin{bmatrix}
* & \lambda & \lambda & \cdot & \\
\mu & * & \cdot & \cdot & \\
\mu & * & \cdot & \lambda & \\
\cdot & \mu & \cdot & * & \\
\cdot & \mu & \cdot & \cdot & *
\end{bmatrix}
\]

From this we can find the transition probabilities: Given that we are in state \(i\) at time \(t\), the probability that we are in state \(j\) at a later time \(t + h\), is the \((i, j)\) element in \(P(t) = \exp(Gh)\). It is possible to obtain analytical expressions for these, for example using \texttt{maple}, but the result is of little use. Instead we display the transition probabilities from state 1 to the 5 states as functions of \(t\):

\[
\begin{align*}
&\text{>> } G2 = [0,1,1,0,0;4,0,0,1,0;4,0,0,0,1;0,4,0,0,0;0,4,0,0,0]; \\
&\text{>> } G2 = G2 - \text{diag}(\text{sum}(G2,2)); \\
&\text{>> } tt = \text{linspace}(0,2,100); \\
&\text{>> } \text{for } i=1:\text{length}(tt), \quad t=tt(i); \\
&\quad \text{Pt}\text{=}\text{expm}(G2*t); \\
&\quad \text{mu}(i,:) = \text{Pt}(1,:); \\
&\text{end} \\
&\text{>> } \text{plot}(tt,mu) \\
&\text{>> } \text{legend}(’p11’,’p12’,’p13’,’p14’,’p15’) \\
&\text{>> } \text{xlabel(’Time’)} \\
&\text{>> } \text{ylabel(’Transition probabilities’)}
\end{align*}
\]

**Question 4:** The easiest to solve for the stationary probabilities using e.g. \texttt{matlab}.

\[
\begin{align*}
&\text{>> } \text{piv2} = \text{null}(G2’); \\
&\text{>> } \text{piv2} = \text{piv2’/sum(piv2)}
\end{align*}
\]
piv2 =

0.6154  0.1846  0.1231  0.0462  0.0308

If you solve the system with maple or by hand you will find

\[ \pi = \frac{1}{65}(40, 12, 8, 3, 2) \]

**Question 5:** The utilization is \(1 - \pi_0 = 5/13 \approx 0.3846\).

This should be compared with the value 0.24 which we found in the previous, without rentals.

**Question 6:** Requests for work that arrive while the system is in state 3 and 4 are not accepted. The probability of this is \(\pi_3 + \pi_4 = 1/13\) or approximately 0.08. Requests for rentals are rejected unless they arrive in state 0, so the probability of rejection is \(1 - \pi_0 = 5/13 \approx 0.38\). Combining, the probability of rejection for a random customer is \(3/13 \approx 0.23\). This should be compared with the 0.05 found above.