IMM - DTU

02407 Stochastic Processes
November 5, 2006
NHH/BFN/UHT

Solution to exercise 27

Question 1: The system can be modeled as a continuous time Markov chain with 3 states:

1. Idle
2. Busy with no order in the book
3. Busy with one order in the book

We find the generator

$$G = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

where $\lambda = 1$ is the arrival rate of requests, and $\mu = 4$ is the rate of completing jobs.

First, we find the steady-state distributions as in general for a birth-death process:

$$\pi_i = c \cdot \rho^i \quad \text{for } i = 1, 2, 3$$

where $\rho = \lambda/\mu$ and $c = (\rho + \rho^2 + \rho^3)^{-1}$ is a normalization constant making the probabilities sum to one, $\pi_i + \pi_2 + \pi_3 = 1$.

Alternatively, we could find the stationary distribution directly numerically e.g. using `matlab`:

```
piv = null(G');
piv = piv/sum(piv);
piv(3)
```

This gives a numeric value of 0.0476 which is 1/21.

Next, we realize that the fraction of customers arriving when the system is in state 2 must equal the fraction of time spent in state 2, and thus the stationary probability of state 2, π_2. This is the PASTA property (Poisson arrivals see time averages). To see this, assume that the system is stationary. The probability of a new customer arriving in the time interval $(t, t+h]$ is independent of which state the system is in at time t. So given that a customer arrives, the probability that the system is in state 2 is still π_2.

So the fraction of customers rejected is $\pi_2 = 1/21$ or approximately 0.05.

Question 2: The fraction of time where the equipment is in use is equal to the probability that the system at a given fixed time is in state 2 or 3, i.e. $\pi_2 + \pi_3 = 5/21$ or approximately 0.24.

Question 3: The states are

0: Idle

1: In use, no order in book

2: Rented, no order in book

3: In use, order in book

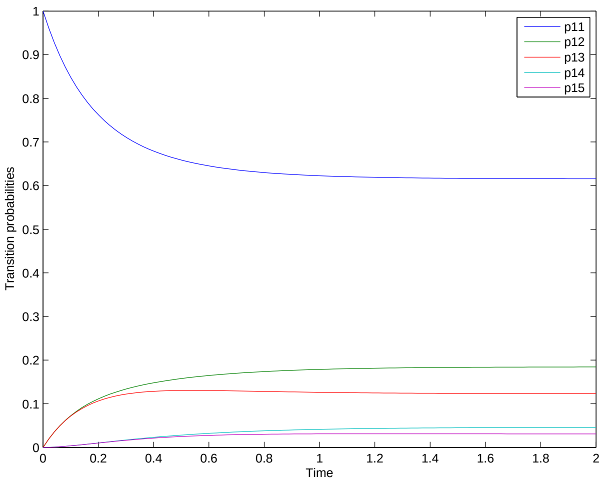
4: Rented, order in book

The generator is

$$\boldsymbol{G} = \begin{bmatrix} \star & \lambda & \lambda & \cdot & \cdot \\ \mu & \star & \cdot & \lambda & \cdot \\ \mu & \cdot & \star & \cdot & \lambda \\ \cdot & \mu & \cdot & \star & \cdot \\ \cdot & \mu & \cdot & \cdot & \star \end{bmatrix}$$

From this we can find the transition probabilities: Given that we are in state i at time t, the probability that we are in state j at a later time $t+h$, is the (i,j) element in $\boldsymbol{P}(t) = \exp(\boldsymbol{G}h)$. It is possible to obtain analytical expressions for these, for example using `maple`, but the result is of little use. In stead we display the transition probabilities from state 1 to the 5 states as functions of t:

```
>> G2 = [0,1,1,0,0;4,0,0,1,0;4,0,0,0,1;0,4,0,0,0;0,4,0,0,0];
>> G2 = G2 - diag(sum(G2,2));
>> tt = linspace(0,2,100);
>> for i=1:length(tt),
     t=tt(i);
     Pt=expm(G2*t);
     mu(i,:) = Pt(1,:);
   end
>> plot(tt,mu)
>> legend('p11','p12','p13','p14','p15')
>> xlabel('Time')
>> ylabel('Transition probabilities')
```



Question 4: The easiest to solve for the stationary probabilities using e.g. `matlab`.

```
>> piv2 = null(G2');
>> piv2 = piv2'/sum(piv2)
```

```
piv2 =

    0.6154    0.1846    0.1231    0.0462    0.0308
```

If you solve the system with `maple` or by hand you will find

$$\pi = \frac{1}{65}(40, 12, 8, 3, 2)$$

Question 5: The utilization is $1 - \pi_0 = 5/13 \approx 0.3846$.

This should be compared with the value 0.24 which we found in the previous, without rentals.

Question 6: Requests for work that arrive while the system is in state 3 and 4 are not accepted. The probability of this is $\pi_3 + \pi_4 = 1/13$ or approximately 0.08. Requests for rentals are rejected unless they arrive in state 0, so the probability of rejection is $1 - \pi_0 = 5/13 \approx 0.38$. Combining, the probability of rejection for a random customer is $3/13 \approx 0.23$. This should be compared with the 0.05 found above.