IMM - DTU

02407 Stochastic Processes
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DAME/dame

Solution for exercise 9.2.5 in Karlin and Pinsky

We choose $P_{kj}(t) = P(D(t) = j|N(0) = k)$ and
$P_j(t) = \sum \pi_k P_{kj}(t) = P(D(t) = j)$
Doing a first step analysis we look first at $k = 0$, in an infinitesimal interval Δt either one new customer arrives and we have to consider P_{1j} or no customer arrives and we look at P_{0j}.

$$P_{0j}(t + \Delta t) = \lambda \Delta t P_{1j}(t) + [1 - \lambda \Delta t] P_{0j}(t) + o(\Delta t)$$

Similar for $k > 0$ either one customer arrives, one customer leaves the system and only $j - 1$ customer have to leave the system or nothing changes.

$$P_{kj} = \mu \Delta t P_{k-1.j-1}(t) + \lambda \Delta t P_{k+1,j}(t) + [1 - (\lambda + \mu)\Delta t] P_{kj}(t) + o(\Delta t)$$

We can use this to derive

$$\begin{aligned}
P_j(t + \Delta t) &= \sum \pi_k (P_{kj}(t + \Delta t)) \\
&= \pi_0 \left(\lambda \Delta t P_{1j}(t) + [1 - \lambda \Delta t] P_{0j}(t) + O(\Delta t)\right) \\
&+ \sum_{k=1}^{\infty} \pi_k \left(\mu \Delta t P_{k-1,j-1}(t) + \lambda \Delta t P_{k+1j}(t) + [1 - (\lambda + \mu)\Delta t] P_{kj}(t) + O(\Delta t)\right) \\
&= \sum_{k=0}^{\infty} \pi_k P_{kj}(t) + \pi_0 \lambda \Delta t P_{1j}(t) - \pi_0 \lambda \Delta t P_{0j}(t) \\
&+ \sum_{k=1}^{\infty} \pi_k \mu \Delta t P_{k-1,j-1}(t) + \sum_{k=1}^{\infty} \pi_k \lambda \Delta t P_{k+1j}(t) \\
&- \sum_{k=1}^{\infty} \pi_k \lambda \Delta t P_{kj}(t) - \sum_{k=1}^{\infty} \pi_k \mu \Delta t P_{kj}(t) + o(h)
\end{aligned}$$

If we use the fact $\pi_k\lambda = \pi_{k+1}\mu$, we can get

$$\begin{aligned}
P_j(t+\Delta t) &= P_j(t) + \sum_{k=1}^{\infty} \pi_k \mu \Delta t P_{k-1,j-1}(t) + \sum_{k=0}^{\infty} \pi_k \lambda \Delta t P_{k+1j}(t) \\
&- \sum_{k=0}^{\infty} \pi_k \lambda \Delta t P_{kj}(t) - \sum_{k=1}^{\infty} \pi_k \mu \Delta t P_{kj}(t) + o(h) \\
&= P_j(t) + \sum_{k=1}^{\infty} \pi_{k-1} \lambda \Delta t P_{k-1,j-1}(t) + \sum_{k=0}^{\infty} \pi_{k+1} \mu \Delta t P_{k+1j}(t) \\
&- \sum_{k=0}^{\infty} \pi_k \lambda \Delta t P_{kj}(t) - \sum_{k=1}^{\infty} \pi_k \mu \Delta t P_{kj}(t) + o(h) \\
&= P_j(t) + \lambda \Delta t P_{j_1}(t) - \lambda \Delta t P_j(t)
\end{aligned}$$

we can use this to get

$$\frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} = \lambda P_{j-1}(t) - \lambda P_j(t) + o(\Delta t)/\Delta t$$

For $\Delta t \to 0$ we get the differential equation of a Poisson process!