IMM - DTU

02407 Stochastic Processes
2012-11-20
DAME/dame

Solution for exercise 8.4.10 in Karlin and Pinsky

In order to show that $X_n = Z(t_n)e^{-rt_n}$ were $Z9t)$ is the geometric Brownian motion, we have to show:
i) $E[|X(t_n)|] < \infty$
ii) $E[X(t_{n+1})|X(t_0), \cdots, X(t_n)] = X(t_n)$
First we have a look at i)

$$\begin{aligned} E[|X(t+n)|] &= E[|Z(t_n)|] \cdot e^{-r \cdot t_n} \\ - &= |z| \cdot e^{\alpha \cdot t_n} \cdot e^{-r \cdot t_n} \\ &= |z| \cdot e^{-(r-\alpha)t_n} \quad \text{given } r = \alpha \\ &= |z| < \infty \end{aligned}$$

Now we can face ii). Since the geometric Brownian motion is a Markov Process, it is enough to show $E[X(t_{n+1})|X(t_n)] = X(t_n)$. Let us assume

$X(t_n) \neq 0$

$$
\begin{aligned}
E[X(t_{n+1})|X(t_n)] &= E\left[X(t_{n+1})\frac{X(t_n)}{X(t_n)}|X(t_n)\right] \\
&= E\left[\frac{X(t_{n+1})}{X(t_n)}|X(t_n)\right]X(t_n) \\
&= E\left[\frac{Z(t_{n+1})e^{-rt_{n+1}}}{Z(t_n)e^{-rt_n}}|X(t_n)\right]X(t_n) \\
&= E\left[e^{(\alpha-\frac{1}{2}\sigma^2)(t_{n+1}-t_n)+\sigma(B(t_{n+1})-B(t_n))}|X(t_n)\right]e^{-r(t_{n+1}-t_n)}X_n \\
&= E\left[e^{\sigma B(t_{n+1})-B(t_n)}|X(t_n)\right]e^{(\alpha-\frac{1}{2}\sigma^2)(t_{n+1}-t_n)}e^{-r(t_{n+1}-t_n)}X_n \\
&= E\left[e^{\sigma B(t_{n+1}-t_n)}|X(t_n)\right]e^{(\alpha-\frac{1}{2}\sigma^2)(t_{n+1}-t_n)}e^{-r(t_{n+1}-t_n)}X_n \\
&= E\left[e^{\sigma B(t_{n+1}-t_n)}|X(t_n)\right]e^{-\frac{1}{2}\sigma^2(t_{n+1}-t_n)}X_n \quad \text{since } \alpha = r \\
&= e^{\frac{1}{2}\sigma^2(t_{n+1}-t_n)}e^{-\frac{1}{2}\sigma^2(t_{n+1}-t_n)}X_n \quad \text{similar to equation 8.50} \\
&= X_n
\end{aligned}
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