IMM - DTU

02407 Stochastic Processes
2011-11-8
DAME/dame

Solution for exercise 8.3.3 in Karlin and Pinsky

Define $B^0(t) := B(t) - tB(1) = B(t) - B(0) - t(B(1) - B(0) \ : \ 0 \leq t \leq 1$ since the increments of a brownien motion are normal distributed we know that $B^0(t)$ is normal distributed.
Knowing this, using the definition of a Gaussion Process on page 398 Karlin and Pinsky, we can argue that $B^0(t)$ is a Gaussian process. We know also, the Brownian motion is a Gaussian process.
In order to show independence of $B^0(t)$ and $B(1)$ it leaves us to show:

$$\begin{aligned} E[B^0(t)B(1)] &= 0 \\ & \text{but} \\ E[B^0(t)B(1)] &= E[B(t)B(1) - tB(1)^2] \\ &= t - t \cdot 1 = 0 \end{aligned}$$

Therefore are $B^0(t)$ and $B(1)$ independent.

0.1 a)

To show that $B^0(t)$ is a Brownian bridge we have only to show that $E[B^0(t) = 0$ as well as $Var[B^0(t) = t(1-t)$, since we already know that it is Normal distributed.

$$\begin{aligned}
E[B^0(t)] &= E[B(t) - tB(1)] \\
&= E[B(t)] - tEB(1) \\
&= 0 - t \cdot 0 = 0 \\
Var[B^0(t)] &= E[\big(B^0(t)\big)^2] - E[B^0(t)]^2 \\
&= E[(B(t) - tB(1))^2 \\
&= E[(B(t))^2 - 2tB(1)B(t) + t^2B(1)] \\
&= t - 2t + t^2 \cdot 1 \\
&= t(1-t)
\end{aligned}$$

After all $B^0(t)$ is a Brownian bridge.

0.2 b)

now we have to calculate the covariance function $Cov[B^0(s)B^0(t)] = E[B^0(s)B^0(t)]$. Without loss of generallity, we can assume $s \leq t$.

$$\begin{aligned}
E[B^0(s)B^0(t)] &= E[(B(s) - sB(1))(B(t) - tB(1))] \\
&= E[B(s)B(t)] - E[sB(1)B(t)] - E[B(s)tB(1)] - E[stB(1)^2] \\
&= s - s \cdot t - s \cdot t - s \cdot t \\
&= s(1-t)
\end{aligned}$$

For the general case we get $Cov[B^0(s)B^0(t)] = mins, t(1 - maxs, t)$