IMM - DTU

02407 Stochastic Processes
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Solution for exercise 8.3.1 in Karlin and Pinsky

We are supposed to consider the random variable $R(t) = \sqrt{B_1(t)^2 + B_2(t)^2}$ with $B_i(t)$ is a standard Brownian motion. Furthermore we know $B_i(t) \sim N(0,t)$. since the Brownian motions are independent we can easily derive the joint density:

$$\begin{aligned} f_{B_1(t),B_2(t)}(x,y) &= f_{B_1(t)} \cdot f_{B_2(t)} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}y^2} \\ &= \frac{1}{2\pi t} e^{-\frac{1}{2t}(x^2+y^2)} \end{aligned}$$

With $r = \sqrt{x^2+y^2}$ we get $r^2 = x^2 + y^2$. Furthermore the circumference of a circle with radius r is $2\pi r$. Since $f_{B_1(t),B_2(t)}(x,y)$ is almost constant om the circle $f_{B_1(t),B_2(t)}(x,y)$ we get

$$\begin{aligned} f_{R(t)}(r) &= f_{B_1(t),B_2(t)}(x,y) \cdot 2\pi r = \frac{1}{2\pi t} e^{-\frac{1}{2t}r^2} \\ &= \frac{r}{t} e^{-\frac{1}{2t}r^2} \end{aligned}$$

Calculating the expectation is then straight forward

$$\begin{aligned} E[R(t)] &= \int_0^\infty r \cdot f_{R(t)}(r) dr \\ &= \int_0^\infty r \cdot \frac{r}{t} e^{-\frac{1}{2t}r^2} dr \\ &= \frac{1}{t} \int_0^\infty r^2 e^{-\frac{1}{2t}r^2} dr \\ &= \frac{\sqrt{2\pi t}}{t} \int_0^\infty r^2 \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}r^2} dr \end{aligned}$$

Due to symmetry we can write $\int_0^\infty r^2 \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}r^2} dr = \frac{1}{2} \int_{-\infty}^\infty r^2 \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}r^2} dr$. The integral on the right hand side is the variance of a $N(0,t)$ distributed random variable and we get

$$
\begin{aligned}
E[R(t)] &= \frac{\sqrt{2\pi t}}{2t} \int_{-\infty}^{\infty} r^2 \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t} r^2}\, dr \\
&= \frac{\sqrt{2\pi t}}{2t} \cdot t = \sqrt{\frac{\pi t}{2}}
\end{aligned}
$$