IMM - DTU

02407 Stochastic Processes
2011-11-8
DAME/dame

Solution for exercise 8.2.4 in Karlin and Pinsky

$$
\begin{aligned}
P(M(t) \geq z, B(t) \leq x) &= P(\exists\ a < t \ with\ B(a) \geq z, B(t) \leq x) \\
&= P(\exists\ a < t \ with\ B(a) \geq z, B(t-a) \leq -z + x) \\
&= P(\exists\ a < t \ with\ B(a) \geq z, B(t-a) \geq z - x) \\
& \quad \text{usingt the reflection principle,} \\
& \quad \text{see Figure 8.4 p.406 Karlin and Pinksy, we get:} \\
&= P(\exists\ a < t \ with\ B(a) \geq z, B(t) \geq 2z - x) \\
&= P(B(t) \geq 2z - x) \\
&= 1 - \Phi\left(\frac{2z - x}{\sqrt{t}}\right)
\end{aligned}
$$

In order to optain the joint density function we need to calculate:

$$
\frac{\delta^2}{\delta x \delta z}\left(P(M(t) \leq z, B(t) \leq x)\right) = \frac{\delta^2}{\delta x \delta z}\left(PB(t) \leq x) - P(M(t) \geq z, B(t) \leq x)\right)
$$

Since $P(B(t) \leq x)$ is independent of z:

$$
\frac{\delta^2}{\delta x \delta z}\left(PB(t) \leq x)\right) = 0
$$

First we differentiate with respect to x

$$
\begin{aligned}
\frac{\delta}{\delta x} P(M(t) \geq z, B(t) \leq x) &= \frac{\delta}{\delta x}\left(1 - \int_{-infty}^{\frac{2z-x}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right) \\
&= -\left(-\frac{1}{\sqrt{t}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{2z-x}{\sqrt{t}}\right)^2} \\
&= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{2z-x}{\sqrt{t}}\right)^2}
\end{aligned}
$$

Now differantiating with respect to z

$$
\begin{aligned}
& \frac{\delta}{\delta z}\left(\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{2z-x}{\sqrt{t}})^2}\right) \\
= & \frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{2z-x}{\sqrt{t}}\right)^2}\cdot\left(-\frac{1}{2}2\frac{2z-x}{\sqrt{t}}\cdot\frac{2}{\sqrt{t}}\right) \\
= & -\frac{2z-x}{t}\frac{2}{\sqrt{t}}\phi\left(\frac{2z-x}{\sqrt{t}}\right)
\end{aligned}
$$

The final result is:

$$
\begin{aligned}
\frac{\delta^2}{\delta x\delta z}\left(P(M(t)\leq z, B(t)\leq x)\right) &= \frac{\delta^2}{\delta x\delta z}\left(PB(t)\leq x) - P(M(t)\geq z, B(t)\leq x)\right) \\
&= \frac{\delta^2}{\delta x\delta z}\left(P(B(t)\leq x)\right) - \frac{\delta^2}{\delta x\delta z}P(M(t)\geq z, B(t)\leq x) \\
&= 0 - \left(-\frac{2z-x}{t}\frac{2}{\sqrt{t}}\phi\left(\frac{2z-x}{\sqrt{t}}\right)\right)
\end{aligned}
$$