IMM - DTU

02407 Stochastic Processes
2011-11-8
DAME/dame

Solution for exercise 8.1.7 in Karlin and Pinsky

For a stochastic process to be a martingal the following conditions have to be fullfilled:
i) $E[|X_n|] < \infty$ for all n
ii) $E[X_{n=1}|X_0, \cdots, X_n] = X_n$ for all n

0.1 a)

i)

$$\begin{aligned} E[|B_n|] &= \int_{-\infty}^{\infty} |x|\phi_n(x)dx \\ &= 2\int_0^{\infty} x\phi_n(x)dx \\ &= 2\int_0^{\infty} \frac{x}{\sqrt{2\pi n}} e^{-\frac{x^2}{n}}dx \\ &= \sqrt{\frac{2n}{\pi}} < \infty \end{aligned}$$

the integeral can be evaluated through the change $y = \frac{x}{\sqrt{n}}$
ii)

$$\begin{aligned} E[B_{n=1}|B_0, \cdots, B_n] &= E[B_{n+1}|B_n] \\ &= E[B_{n+1} - B_n + B_n|B_n] \\ &= E[B_{n+1} - B_n|B_n] + E[B_n|B_n] \\ &= 0 + B_n = B_n \end{aligned}$$

0.2 b)

i) We know $B_n \geq 0$ and $-n \leq 0$ and therefore $E[|B_n^2 - n| = |E[B_n^2 - n]|$. We use this and optain:

$$E[|B_n^2 - n|] = |E[B_n^2 - n]| = |E[B_n^2] - n| = |n - n| = 0 < \infty$$

ii)

$$\begin{aligned}
E[B_{n+1}^2 - n - 1|B_n^2 - n] &= E[B_{n+1}^2|B_n^2 - n] - (n+1) \\
&= E[(B_{n+1} - B_n + B_n)^2|B_n^2 - n] - (n+1) \\
&= E[(B_{n+1} - B_n)^2 + 2(B_{n+1} - B_n)B_n + B_n^2|B_n^2 - n] - (n+1) \\
&= E[(B_{n+1} - B_n)^2|B_n^2 - n] + 2E[(B_{n+1} - B_n)B_n|B_n^2 - n] \\
&+ E[B_n^2|B_n^2 - n] - (n-1) \\
&= 1 + 2 \cdot 0B_n^2 - (n-1) = B_n^2 - n
\end{aligned}$$