IMM - DTU

02407 Stochastic Processes
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Solution for exercise 7.5.4 in Karlin and Pinsky

Let Z_1 and Z_2 be the lifetime of light bulb one and two.
With $Y_1 = min(Z_1, Z_2)$ and $X_1 = max(Z_1, Z_2)$ we can use equation 7.27 from page 373 to answer the question.

0.1 a)

We know, the minimum of two $exp(\lambda)$ distributed Variables, is $exp(2\lambda)$ distributed and the maximum of two $exp(\lambda)$ is generalised Erlang-2[1] distributed

$$\begin{aligned}\lim_{t\to\infty} p(t) &= \frac{E[Y_1]}{E[X_1]} \\ &= \frac{\frac{1}{2\lambda}}{\frac{1}{2\lambda}+\frac{1}{\lambda}} \\ &= \frac{1}{3}\end{aligned}$$

So the lazy professor sits one third of the time in an office with two lit light bulbs and two thirds of the time in a half lit office.

[1]A generalised Erlang distribution is the distribution of a sum of a independent exponential random variables with possibly different intensites

0.2 b)

This time we are dealing with independent uniform distributions. We know $F_{Z_1}(x) = x$ as well as $F_{Z_2}(x) = x$. Furthermore we get

$$\begin{aligned}
F_{Y_1}(x) &= P(\min(Z_1, Z_2) \leq x) = 1 - P(\min(Z_1, Z_2) > x) \\
&= 1 - P(Z_1 > x) \cdot P(Z_2 > x) = 1 - (1-x)^2 = 2x - x^2 \\
\Rightarrow\ f_{Y_1}(x) &= \frac{d\ F_{Y_1}(x)}{d\ x} = 2 - 2x \\
\Rightarrow\ E[Y_1] &= \int_0^1 x \cdot f_{Y_1}(x) dx = \int_0^1 2x - 2x^2 dx \\
&= [x^2 \frac{2}{3} x^3]_0^1 = \frac{1}{3}
\end{aligned}$$

and

$$\begin{aligned}
F_{X_1}(x) &= P(\max(Z_1, Z_2) \leq x) = P(Z_1 \leq x) \cdot P(Z_2 \leq x) = x^2 \\
\Rightarrow f_{X_1}(x) = 2x & \\
\Rightarrow E[X_1] &= \int_0^1 2x^2 dx = \frac{2}{3} \\
\Rightarrow \lim_{t \to \infty} p(t) &= \frac{E[Y_1]}{E[X_1]} = \frac{1}{2}
\end{aligned}$$

In this case the office is half lit, half of the time.