IMM - DTU

02407 Stochastic Processes
2011-9-27
DAME/dame

Solution for exercise 6.2.6 in Karlin and Pinsky

0.1 a)

$$\begin{aligned} E[T] &= E[W_N] = E[\sum_{i=1}^{N} S_i] \\ &= \sum_{i=1}^{N} E[S_i] \\ &= \sum_{i=1}^{N} \frac{1}{i \cdot \alpha} \\ &= \frac{1}{\alpha}[\frac{1}{1} + \cdots + \frac{1}{N}] \end{aligned}$$

0.2 b)

First we consider

$$\begin{aligned} y : &= 1 - e^{-\alpha t} \\ \Rightarrow \frac{dy}{dt} &= \alpha \cdot e^{-\alpha t} \\ \Rightarrow dt &= \frac{dy}{\alpha(1-y)} \end{aligned}$$

As well as $y(0) = 0$ and $y(\infty) = 1$
now we calculate

$$\begin{aligned}
E[T] &= \int_0^\infty P(T > t)dt = \int_0^\infty [1 - F_T(t)]dt \\
&= \int_0^\infty 1 - (1 - e^{-\alpha t})^N dt \\
&= \int_0^1 \frac{1 - y^N}{\alpha(1 - y)} dy \\
&= \frac{1}{\alpha} \int_0^1 \frac{1 - y^n}{1 - y} dy \\
&= \frac{1}{\alpha} \int_0^1 \frac{(1 - y)\sum_{i=0}^{N-1} y^i}{1 - y} \\
&= \frac{1}{\alpha} [\sum_{i=0}^{N-1} \frac{1}{i+1} y^{i+1}]_0^1 \\
&= \frac{1}{\alpha} \sum_{i=1}^{N} \frac{1}{i} \\
&= \frac{1}{\alpha} [\frac{1}{N} + \cdots + \frac{1}{1}]
\end{aligned}$$