IMM - DTU

02407 Stochastic Processes
2012-10-05
DAME/dame

Solution for exercise 6.1.2 in Karlin and Pinsky

From Chapter 6.1 we know

$$\begin{aligned} P_n(t) &= \prod_{i=0}^{n-1} \lambda_i [\sum_{k=0}^{n} B_{k,n} e^{\lambda_k t}] \\ B_{k,n} &= \frac{1}{\prod_{i=0\ i\neq k}^{n} (\lambda_i - \lambda_k)} \end{aligned}$$

and with $\lambda_k = \alpha + k\beta$ we optain

$$\begin{aligned} B_{k,n} &= \frac{(-1)^k}{\beta^n (n-k)! k!} \\ \Rightarrow\ P_n(t) &= \prod_{i=0}^{n-1} (\alpha + i\cdot\beta) \cdot \sum_{k=0}^{n} \frac{(-1)^k}{\beta^n (n-k)! k!} \cdot e^{-(\alpha+k\beta)t} \\ &= \prod_{i=0}^{n-1} (\alpha + i\cdot\beta) \cdot e^{-\alpha t} \beta^{-n} \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! k!} \cdot e^{-k\beta t} \\ &= \prod_{i=0}^{n-1} (\alpha + i\cdot\beta) \cdot e^{-\alpha t} \beta^{-n} \cdot \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)! k!} \cdot (-e^{-\beta t})^k \\ &= \prod_{i=0}^{n-1} (\alpha + i\cdot\beta) \cdot e^{-\alpha t} \beta^{-n} \cdot \frac{1}{n!} (1 - e^{-\beta t})^n \\ &= \frac{\beta^n \Gamma(n + \frac{\alpha}{\beta})}{\Gamma(\frac{\alpha}{\beta})} \cdot e^{-\alpha t} \beta^{-n} \cdot \frac{1}{n!} (1 - e^{-\beta t})^n \\ &= \frac{\Gamma(n + \frac{\alpha}{\beta})}{\Gamma(\frac{\alpha}{\beta}) n!} \cdot e^{-\alpha t} (1 - e^{-\beta t})^n \end{aligned}$$

If we choose $\alpha = \beta$ we optain the same result as in a yule process up to $n+1$ indiviuals by reproduction.
Instead of the Gamma function we could as well use generalized binomial coefficens.