DTU Compute

02407 Stochastic Processes
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Solution for problem 5.6.3 in Pinsky & Karlin

We know the shocks arrive in a Poisson process. Let $N(t)$ be the number of shocks until time t. Let $X(t) = Y_1 + \ldots + Y_{N(t)}$ be the accumulated damage. Furthermore we define $G^{(n)}(a) = P(Y_1 + \ldots + Y_n < a)$, $G^{(0)}(a) = 1$ and $T = \min\{t \geq 0 : X(t) \geq a\}$. We know then $T > t \Leftrightarrow X(t) < a$ and therefore

$$P(T > t) = P\left(X(t) < a\right) = \sum_{n=0}^{\infty} P\left(N(t) = n\right) G^{(n)}(a)$$

With this we can calculate (see also page 266 in KP)

$$\begin{aligned}
E[T] &= \int_0^\infty P(T > t) dt \\
&= \int_0^\infty \sum_{n=0}^{\infty} P\left(N(t) = n\right) G^{(n)}(a)\ dt \\
&= \sum_{n=0}^{\infty} \left(\int_0^\infty P\left(N(t) = n\right)\ dt \right)\ G^{(n)}(a) \\
&= \sum_{n=0}^{\infty} \left(\int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t}\ dt \right)\ G^{(n)}(a) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{\lambda n!} \int_0^\infty (\lambda t)^n e^{-\lambda t} \lambda dt \right)\ G^{(n)}(a) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{\lambda n!} \int_0^\infty u^n e^{-u} du \right)\ G^{(n)}(a) \qquad \text{(substituting } u = \lambda t\text{)} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\lambda n!}\ G^{(n)}(a) \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} G^{(n)}(a)
\end{aligned}$$

The first equality is an alternative formula for the expected value of a non-negative random variable. Third equality holds because all summands are non-negative.

Because Y_i follows a geometric distributed, the sum $Y_1 + \cdots + Y_n$ follows a negative binomial distribution with parameters n and $1-p$, i.e.

$$P(X(t) = k|N(t) = n) = P(Y_1 + \cdots + Y_n = k) = \binom{k+n-1}{k}(1-p)^k p^n$$

Therefore

$$G^{(n)}(a) = \sum_{k=0}^{a-1} \binom{k+n-1}{k}(1-p)^k p^n, \quad n \geq 1$$

and

$$\begin{aligned}
E[T] &= \frac{1}{\lambda}\sum_{n=0}^{\infty} G^{(n)}(a) \\
&= \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty}\sum_{k=0}^{a-1}\binom{k+n-1}{k}(1-p)^k p^n\right) \\
&= \frac{1}{\lambda}\left(1 + \sum_{k=0}^{a-1} p(1-p)^k \sum_{n=1}^{\infty}\binom{k+n-1}{n-1} p^{n-1}\right) \\
&= \frac{1}{\lambda}\left(1 + \sum_{k=0}^{a-1} p(1-p)^k \sum_{n=0}^{\infty}\binom{k+n}{n} p^{n}\right) \\
&= \frac{1}{\lambda}\left(1 + \sum_{k=0}^{a-1} p(1-p)^k (1-p)^{-(k+1)}\right) && \text{(binomial theorem)} \\
&= \frac{1}{\lambda}\left(1 + \sum_{k=0}^{a-1} p(1-p)^{-1}\right) \\
&= \frac{1}{\lambda}\left(1 + \frac{ap}{1-p}\right)
\end{aligned}$$