Solution for problem 5.4.8 in Pinsky & Karlin

The derivation is similar to that on page 249/250 in KP. Let \( \mu_\xi = E[\xi_i] \) and let \( U_1, \ldots, U_n \) be independent and uniformly distributed in \((0,t]\).

\[
E[Z(t)] = E \left[ \sum_{k=1}^{N(t)} \theta_k(t) \right]
\]

\[
= \sum_{n=1}^{\infty} P(N(t) = n) E \left[ \sum_{k=1}^{N(t)} \theta_k(t) \big| N(t) = n \right]
\]

\[
= \sum_{n=1}^{\infty} P(N(t) = n) E \left[ \sum_{k=1}^{N(t)} \xi_k e^{-\alpha(t-W_k)} \big| N(t) = n \right]
\]

\[
= \sum_{n=1}^{\infty} P(N(t) = n) E \left[ \sum_{k=1}^{n} \xi_k e^{-\alpha(t-U_k)} \right]
\]

(by symmetry of \(U_k \) and \(t-U_k\))

\[
= \sum_{n=1}^{\infty} P(N(t) = n) \sum_{k=1}^{n} E[\xi_k] E[e^{-\alpha U_k}]
\]

(W and \(\xi\) are independent)

\[
= \sum_{n=1}^{\infty} P(N(t) = n) \mu_\xi n E[e^{-\alpha U_1}]
\]

\[
= \mu_\xi E[e^{-\alpha U_1}] E[N(t)]
\]

\[
= \mu_\xi \left( \int_0^t e^{-\alpha u} \frac{1}{t} du \right) \lambda t
\]

\[
= \frac{\mu_\xi \lambda}{\alpha} (1 - e^{-\alpha t})
\]

This is also a generalization of the shot noise process in KP page 253/254.