DTU Compute

02407 Stochastic Processes
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Solution for problem 5.4.8 in Pinsky & Karlin

The derivation is similar to that on page 249/250 in KP. Let $\mu_\xi = E[\xi_i]$ and let $U_1, \ldots, U_n$ be independent and uniformly distributed in $(0, t]$.

$$
\begin{aligned}
E\left[Z(t)\right] &= E\left[\sum_{k=1}^{N(t)} \theta_k(t)\right] \\
&= \sum_{n=1}^{\infty} P(N(t)=n) E\left[\sum_{k=1}^{N(t)} \theta_k(t) | N(t)=n\right] \\
&= \sum_{n=1}^{\infty} P(N(t)=n) E\left[\sum_{k=1}^{N(t)} \xi_k e^{-\alpha(t-W_k)} | N(t)=n\right] \\
&= \sum_{n=1}^{\infty} P(N(t)=n) E\left[\sum_{k=1}^{n} \xi_k e^{-\alpha(t-U_k)}\right] && \text{(Theorem 5.7)} \\
&= \sum_{n=1}^{\infty} P(N(t)=n) E\left[\sum_{k=1}^{n} \xi_k e^{-\alpha U_k}\right] && \text{(by symmetry of } U_k \text{ and } t-U_k) \\
&= \sum_{n=1}^{\infty} P(N(t)=n) \sum_{k=1}^{n} E[\xi_k] E\left[e^{-\alpha U_k}\right] && (W \text{ and } \xi \text{ are independent}) \\
&= \sum_{n=1}^{\infty} P(N(t)=n) \mu_\xi n E\left[e^{-\alpha U_1}\right] \\
&= \mu_\xi E\left[e^{-\alpha U_1}\right] E[N(t)] \\
&= \mu_\xi \left(\int_0^t e^{-\alpha u} \frac{1}{t} du\right) \lambda t \\
&= \frac{\mu_\xi \lambda}{\alpha}(1 - e^{-\alpha t})
\end{aligned}
$$

This is also a generalization of the shot noise process in KP page 253/254.