IMM - DTU

02407 Stochastic Processes
2011-9-30
DAME/dame

Solution for exercise 5.3.9 in Karlin and Pinsky

0.1 a)

We have to calculate the minimum of two exponential distributed waiting times until the first event. This is:

$$\begin{aligned} P(X_1 < X_2) &= E[P(X_1 < X_2 | X_1) \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\ &= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)x} dx \\ &= \lambda_1 \cdot \frac{1}{\lambda_1 + \lambda_2} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

0.2 b)

This time we consider the waiting time until the second event.

$$\begin{aligned} P(X_1 < X_2) &= E[P(X_1 < X_2 | X_1)] \\ &= \int_0^\infty (e^{-\lambda_2 x} + \lambda_2 x e^{-\lambda_2 x}) \lambda_1^2 x e^{-\lambda_1 x} dx \\ &= \lambda_1^2 \int_0^\infty x e^{-(\lambda_2+\lambda_1)x} dx + \lambda_1^2 \lambda_2 \int_0^\infty x^2 e^{-(\lambda_2+\lambda_1)} dx \\ &= \frac{\lambda_1^2}{\lambda_1 + \lambda_2} \int_0^\infty x(\lambda_2 + \lambda_1) e^{-(\lambda_2+\lambda_1)x} dx + \frac{\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^2} \int_0^\infty x^2 (\lambda_2 + \lambda_1) e^{-(\lambda_2+\lambda_1)} dx \\ &= \frac{\lambda_1^2}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2 + \lambda_1} + \frac{\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^2} \frac{2}{\lambda_2 + \lambda_1} \\ &= \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2} + 2 \frac{\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)^3} \end{aligned}$$