IMM - DTU

02407 Stochastic Processes
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Solution for problem 5.2.6 in Karlin and Pinsky

We see that:

$$P(X_i = k) \quad = \quad \binom{N}{k}(\frac{1}{M})^k(1 - \frac{1}{M})^{N-k}$$

We know from chapter one (page 23)

$$\begin{aligned}
\lim_{N\to\infty} P(X_i = k) &= \lim_{N\to\infty} \binom{N}{k}\left(\frac{1}{M}\right)^k \left(1 - \frac{1}{M}\right)^{N-k} \\
&= \lim_{N\to\infty} \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \\
&= \lim_{N\to\infty} N(N-1)\cdots(N-k+1)\frac{\lambda^k}{N^k k!}\left(1 - \frac{\lambda}{N}\right)^{N-k} \\
&= \lim_{N\to\infty} 1(1 - \frac{1}{N})\cdots(1 - \frac{k-1}{N}) \cdot \frac{\lambda^k(1-\frac{\lambda}{N})^N}{k!(1-\frac{\lambda}{N})^k} \\
&= 1 \cdot \frac{\lambda^k \cdot e^{-\lambda}}{k!(1-0)^k} \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}$$

To prove independence for the limit we show that $\lim_{N\to\infty} P(X_i = k | X_j = l) = \lim_{N\to\infty} P(X_i = k)$

$$\begin{aligned}
\lim_{N\to\infty} P(X_i = k|x_j = l) &= \lim_{N\to\infty} \binom{N-l}{k} \left(\frac{1}{M}\right)^k \left(1 - \frac{1}{M}\right)^{N-l-k} \\
&= \lim_{N\to\infty} \frac{(N-l)!}{k!(N-l-k)!} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-l-k} \\
&= \lim_{N\to\infty} (N-l)(N-l-1)\cdots(N-l-k+1)\frac{\lambda^k}{N^k k!} \left(1 - \frac{\lambda}{N}\right)^{N-l-k} \\
&= \lim_{N\to\infty} \left(1 - \frac{l}{N}\right)\left(1 - \frac{l+1}{N}\right)\cdots\left(1 - \frac{l+k-1}{N}\right) \cdot \frac{\lambda^k (1-\frac{\lambda}{N})^N}{k!(1-\frac{\lambda}{N})^{l+k}} \\
&= 1 \cdot \frac{\lambda^k \cdot e^{-\lambda}}{k!(1-0)^k} \\
&= \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lim_{N\to\infty} P(X_i = k)
\end{aligned}$$

In the limit the fraction of locations that have two or more accounts assigned to them are:

$$\begin{aligned}
\lim_{N\to\infty} P(X_i > 1) &= \lim_{N\to\infty} 1 - P(X_i \leq 1) \\
&= 1 - \lim_{N\to\infty} \{P(X_i = 0) + P(X_i = 1)\} \\
&= 1 - e^{-\lambda} - \lambda e^{-\lambda}
\end{aligned}$$