IMM - DTU

02407 Stochastic Processes
2011-9-14
DAME/dame

Solution for exercise 4.1.12 in Karlin and Pinsky

0.1 a)

It has to be shown that $P^n = Q^n + \Pi$ with $Q = P - \Pi$.
Since Π has the stationary distribution as row vectors and P is a stochastic matrix the row sums are equal to one.
Therefor $\Pi \cdot P = \Pi$, $P \cdot \Pi = \Pi$ and $\Pi \cdot \Pi = \Pi$
Furthermore we use that $P^n = Q^n + \Pi \;\Leftrightarrow\; P^n - \Pi = Q^n$

$$\begin{aligned} Q^2 &= (P - \Pi)^2 \\ &= P^2 - P\Pi - \Pi P + \Pi^2 \\ &= P^2 - \Pi - \Pi + \Pi \\ &= P^2 - \Pi \end{aligned}$$

Assuming $Q^{n-1} = P^{n-1} - \Pi$ the equation has only to be proofen for Q^n

$$\begin{aligned} Q^n &= Q^{n-1}Q \\ &= (P^{n-1} - \Pi)(P - \Pi) \\ &= P^n - P^{n-1}\Pi - \Pi P + \Pi^2 \\ &= P^n - \Pi - \Pi + \Pi \\ &= P^n - \Pi \\ &\Leftrightarrow P^n = Q^n + \Pi \end{aligned}$$

0.2 b)

With $\pi = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ we get:

$$
\begin{aligned}
Q &= \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \\
Q^n &= \begin{pmatrix} \frac{1}{4 \cdot 2^{n-1}} & 0 & -\frac{1}{4 \cdot 2^{n-1}} \\ 0 & 0 & 0 \\ -\frac{1}{4 \cdot 2^{n-1}} & 0 & \frac{1}{4 \cdot 2^{n-1}} \end{pmatrix} \\
P^n &= \begin{pmatrix} \frac{1}{4 \cdot 2^{n-1}} + \frac{1}{4} & \frac{1}{2} & -\frac{1}{4 \cdot 2^{n-1}} + \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4 \cdot 2^{n-1}} + \frac{1}{4} & \frac{1}{2} & \frac{1}{4 \cdot 2^{n-1}} + \frac{1}{4} \end{pmatrix}
\end{aligned}
$$