Solution for problem 3.9.4 in Karlin and Pinsky

\[ \phi(s) = 1 - p(1 - s)^\beta = \sum_{k=0}^{\infty} p_k s^k \] is a probability generating function if the coefficients in the power series expansion \( p_k \) can be interpreted as probabilities \( p_k \) that constitute a discrete distribution. This is true if \( p_k \geq 0 \) and \( \sum_k p_k = 1 \).

First we find the \( k \)'th derivative of \( \phi(s) \) which is

\[
\frac{d^k}{ds^k} \phi(s) = (-1)^{k+1} \beta(\beta - 1) \cdots (\beta - k + 1)p(1 - s)^{\beta-k}
\]

Then we use equation (3.104) in KP to write up the probabilities \( p_k \). Note that \( p_0 = \phi(0) = 1 - p \)

\[
p_k = \left. \frac{1}{k!} \frac{d^k}{ds^k} \phi(s) \right|_{s=0} = \left. \frac{1}{k!} (-1)^{k+1} \beta(\beta - 1) \cdots (\beta - k + 1)p \right|_{s=0} = \frac{p \beta (1 - \beta) \cdots (k - 1 - \beta)}{k!} \quad k = 1, 2, \ldots
\]

Because \( 0 < p, \beta < 1 \), every factor in the last expression is in fact positive, hence \( p_k > 0 \) for all \( k \geq 0 \). To show that the sum of probabilities is equal to one we first use Raabe’s test (https://en.wikipedia.org/wiki/Ratio_test#Raabe%27s_test), which is an extension of the ratio test for the convergence of a series. This tells us that if

\[
\lim_{k \to \infty} k \left( \frac{a_k}{a_{k+1}} - 1 \right) > 1
\]

then the series \( \sum_{k=1}^{\infty} a_k \) converges. Inserting \( p_k \) we get

\[
\lim_{k \to \infty} k \left( \frac{p_k}{p_{k+1}} - 1 \right) = \lim_{k \to \infty} k \left( \frac{p \beta (1 - \beta) \cdots (k - 1 - \beta)}{p \beta (1 - \beta) \cdots (k - \beta)} - 1 \right) = \lim_{k \to \infty} k \left( \frac{k + 1}{k - \beta} - 1 \right) = 1 + \beta > 1,
\]

so \( \sum_{k=1}^{\infty} p_k \) converges and of course \( \sum_{k=0}^{\infty} p_k \) also converges. From the definition of the binomial coefficient (KP p.44) and equation (2) we see that \( \phi(s) \) is actually a binomial series,
\[ \phi(s) = \sum_{k=0}^{\infty} p_k s^k \]  

(4)

\[ \phi(s) = p_0 + \sum_{k=1}^{\infty} \frac{1}{k!} (-1)^{k+1} \beta(\beta - 1) \cdots (\beta - k + 1) p s^k \]  

(5)

\[ \phi(s) = 1 - p - p \sum_{k=1}^{\infty} (-1)^k \binom{\beta}{k} s^k \]  

(6)

\[ \phi(s) = 1 + p \sum_{k=0}^{\infty} \binom{\beta}{k} (-1)^k s^k \]  

(7)

From the binomial theorem ((1.70) p.45 in KP) we have

\[ \sum_{k=0}^{\infty} \binom{\beta}{k} (-1)^k s^k = (1 - s)^\beta, \quad -1 < s < 1, \]  

(8)

i.e. the convergence radius of this series is 1. Because \( \sum_{k=0}^{\infty} p_k \) converges and because of (4)-(7), we know that (8) must also be convergent for \( s = 1 \). We then use Abel’s theorem (https://en.wikipedia.org/wiki/Abel%27s_theorem) and get

\[ \sum_{k=0}^{\infty} \binom{\beta}{k} (-1)^k = \lim_{s \to 1} (1 - s)^\beta = 0 \]

Finally we have

\[ \sum_{k=0}^{\infty} p_k = 1 + p \sum_{k=0}^{\infty} \binom{\beta}{k} (-1)^k = 1 + p0 = 1, \]

and we can conclude that \( \phi \) is a probability generating function.

The iterates \( \phi_n(s) \) are found by mathematical induction

\[ \phi_{n+1}(s) = \phi_n(\phi(s)) = \phi(\phi_n(s)) \]

for \( n = 1 \)

\[ \phi_2 = \phi(\phi(s)) \]

\[ = 1 - p(1 - \phi(s))^{\beta} \]

\[ = 1 - p(1 - 1 + p(1 - s)^{\beta})^{\beta} \]

\[ = 1 - p[p(1 - s)^{\beta}]^{\beta} \]

\[ = 1 - p^{1+\beta}(1 - s)^{\beta^2} \]
Assuming the equation holds for all $n$, we need only to prove it holds for $n+1$

$$
\phi_{n+1} = \phi_n(\phi(s))
$$

$$
= 1 - p^{1+\beta+\ldots+\beta^{n-1}} (1 - \phi(s))^{\beta^n}
$$

$$
= 1 - p^{1+\beta+\ldots+\beta^{n-1}} (1 - 1 - p(1 - s)\beta^n)
$$

$$
= 1 - p^{1+\beta+\ldots+\beta^{n-1}+\beta^n ((1 - s))^{\beta^{n+1}}}
$$