DTU Compute

02407 Stochastic Processes
10 Sep 2018
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Solution for problem 3.9.4 in Karlin and Pinsky

$\phi(s) = 1 - p(1-s)^\beta = \sum_{k=0}^{\infty} p_k s^k$ is a probability generating function if the coefficients in the power series expansion p_k can be interpreted as probabilities p_k that constitute a discrete distribution. This is true if $p_k \geq 0$ and $\sum_k p_k = 1$. First we find the k'th derivative of $\phi(s)$ which is

$$\frac{d^k}{ds^k}\phi(s) = (-1)^{k+1}\beta(\beta-1)\cdots(\beta-k+1)p(1-s)^{\beta-k}$$

Then we use equation (3.104) in KP to write up the probabilities p_k. Note that $p_0 = \phi(0) = 1-p$

$$p_k = \frac{1}{k!}\frac{d^k}{ds^k}\phi(s)\bigg|_{s=0} \tag{1}$$

$$= \frac{1}{k!}(-1)^{k+1}\beta(\beta-1)\cdots(\beta-k+1)p \tag{2}$$

$$= \frac{p\beta(1-\beta)\cdots(k-1-\beta)}{k!} \quad k = 1, 2, \ldots \tag{3}$$

Because $0 < p, \beta < 1$, every factor in the last expression is in fact positive, hence $p_k > 0$ for all $k \geq 0$. To show that the sum of probabilities is equal to one we first use Raabe's test (`https://en.wikipedia.org/wiki/Ratio_test#Raabe%27s_test`), which is an extension of the ratio test for the convergence of a series. This tells us that if

$$\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) > 1$$

then the series $\sum_{k=1}^{\infty} a_k$ converges. Inserting p_k we get

$$\begin{aligned}
\lim_{k\to\infty} k\left(\frac{p_k}{p_{k+1}} - 1\right) &= \lim_{k\to\infty} k\left(\frac{p\beta(1-\beta)\cdots(k-1-\beta)}{k!}\frac{(k+1)!}{p\beta(1-\beta)\cdots(k-\beta)} - 1\right) \\
&= \lim_{k\to\infty} k\left(\frac{k+1}{k-\beta} - 1\right) \\
&= \lim_{k\to\infty} k\left(\frac{1+\beta}{k-\beta}\right) \\
&= 1 + \beta > 1,
\end{aligned}$$

so $\sum_{k=1}^{\infty} p_k$ converges and of course $\sum_{k=0}^{\infty} p_k$ also converges. From the definition of the binomial coefficient (KP p.44) and equation (2) we see that $\phi(s)$ is actually a binomial series,

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k \tag{4}$$

$$= p_0 + \sum_{k=1}^{\infty} \frac{1}{k!}(-1)^{k+1}\beta(\beta-1)\cdots(\beta-k+1)ps^k \tag{5}$$

$$= 1 - p - p\sum_{k=1}^{\infty}(-1)^k \binom{\beta}{k} s^k \tag{6}$$

$$= 1 + p\sum_{k=0}^{\infty}\binom{\beta}{k}(-1)^k s^k \tag{7}$$

From the binomial theorem ((1.70) p.45 in KP) we have

$$\sum_{k=0}^{\infty}\binom{\beta}{k}(-1)^k s^k = (1-s)^\beta, \quad -1 < s < 1, \tag{8}$$

i.e. the convergence radius of this series is 1. Because $\sum_{k=0}^{\infty} p_k$ converges and because of (4)-(7), we know that (8) must also be convergent for $s = 1$. We then use Abel's theorem (`https://en.wikipedia.org/wiki/Abel%27s_theorem`) and get

$$\sum_{k=0}^{\infty}\binom{\beta}{k}(-1)^k = \lim_{s\to 1-}(1-s)^\beta = 0$$

Finally we have

$$\sum_{k=0}^{\infty} p_k = 1 + p\sum_{k=0}^{\infty}\binom{\beta}{k}(-1)^k = 1 + p0 = 1,$$

and we can conclude that ϕ is a probability generating function.

The iterates $\phi_n(s)$ are found by mathematical induction

$$\phi_{n+1}(s) \quad = \quad \phi_n(\phi(s)) = \phi(\phi_n(s))$$

for $n = 1$

$$\begin{aligned}
\phi_2 &= \phi(\phi(s)) \\
&= 1 - p(1-\phi(s))^\beta \\
&= 1 - p(1 - 1 + p(1-s)^\beta)^\beta \\
&= 1 - p[p(1-s)^\beta]^\beta \\
&= 1 - p^{1+\beta}(1-s)^{\beta^2}
\end{aligned}$$

Assuming the equation holds for all n, we need only to proof it holds for n+1

$$\begin{aligned}
\phi_{n+1} &= \phi_n(\phi(s)) \\
&= 1 - p^{1+\beta+\cdots+\beta^{n-1}}(1-\phi(s))^{\beta^n} \\
&= 1 - p^{1+\beta+\cdots+\beta^{n-1}}(1-1-p(1-s)^{\beta})^{\beta^n} \\
&= 1 - p^{1+\beta+\cdots+\beta^{n-1}+\beta^n}((1-s))^{\beta^{n+1}}
\end{aligned}$$