DTU Compute

02407 Stochastic Processes
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Solution for problem 3.8.3 in Karlin and Pinsky

As this is a very simple branching process, the solution might also be derived in a simpler manner. This solution uses the notation from section 3.8.3.

a) First, consider the same problem, but where the process stops if the first child is a girl. Then we have a branching process where in each generation n, the process dies out with probability $p_0 = P(girl) = 1/2$ and generates one offspring with probability $p_1 = P(boy) = 1/2$. Considering the extinction probabilities given in (3.100) and (3.101) we have

$$u_n = P(N \leq n) = p_0 + p_1 u_{n-1}$$

Since $u_0 = 0$ and assuming that $u_{n-1} = 1 - (1/2)^{n-1}$ we get by induction on n that

$$\begin{aligned} u_n &= p_0 + p_1 u_{n-1} \\ &= 1/2 + 1/2(1 - (1/2)^{n-1}) \\ &= 1 - (1/2)^n, \quad n = 0, 1, \ldots \end{aligned}$$

The probability of having n children is then given by

$$\begin{aligned} P(N = n) &= P(N \leq n) - P(N \leq n-1) = u_n - u_{n-1} \\ &= (1 - (1/2)^n) - (1 - (1/2)^{n-1}) \\ &= (1 - 1/2)(1/2)^{n-1} = (1/2)^n \end{aligned}$$

Consider the original process and let X be the number of children. Because of the special case that the first child is a girl, the distrubution is as follows:

$$\begin{aligned} P(X = k) &= \begin{cases} 0 & k = 0,\ k = 1 \\ P(N = 1) + P(N = 2) & k = 2 \\ P(N = k) & k \geq 3 \end{cases} \\ &= \begin{cases} 0 & k = 0,\ k = 1 \\ \frac{3}{4} & k = 2 \\ (\frac{1}{2})^k & k \geq 3 \end{cases} \end{aligned}$$

b) Let X_m denote the number of boys. This is the same process except that the number of boys is equal to the length of the process minus 1, with the two special cases of 0 and 1 boy.

$$P(X_m = k) = \begin{cases} P(\text{two girls}) & k = 0 \\ P(\text{girl then boy or boy then girl}) & k = 1 \\ P(N = k+1) & k \geq 2 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & k = 0 \\ \frac{1}{2} & k = 1 \\ (1/2)^{k+1} & k \geq 2 \end{cases}$$