Birth and Death Processes

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Birth and Death Processes

Today:

▶ Birth processes
▶ Yule process
▶ Death processes
▶ Birth and death processes

Next week

▶ Limiting behaviour of birth and death processes
▶ Birth and death processes with absorbing states
▶ Finite state continuous time Markov chains

Two weeks from now

▶ Renewal phenomena

Birth and Death Processes

Poisson postulates

1. \( \Pr\{X(t + h) - X(t) = 1 | X(t) = x\} = \lambda h + o(h) \)
2. \( \Pr\{X(t + h) - X(t) = 0 | X(t) = x\} = 1 - \lambda h + o(h) \)
3. \( X(0) = 0 \)

Where

\[
\lim_{h \to 0+} \frac{\Pr\{X(t + h) - X(t) = 1 | X(t) = x\}}{h} = \lambda + o(h)
\]
Birth Process Postulates

i \( P\{X(t+h) - X(t) = 1 | X(t) = k\} = \lambda_k h + o(h) \)

ii \( P\{X(t+h) - X(t) = 0 | X(t) = k\} = 1 - \lambda_k h + o(h) \)

iii \( X(0) = 0 \) (not essential, typically used for convenience)

We define \( P_n(t) = P\{X(t) = n | X(0) = 0\} \)

Sojourn times

Define \( S_k \) as the time between the \( k \)th and \((k+1)\)st birth

\[
P_n(t) = P\left\{ \sum_{k=0}^{n-1} S_k \leq t < \sum_{k=0}^{n} S_k \right\}
\]

where \( S_i \sim \text{exp}(\lambda_i) \).

With \( W_k = \sum_{i=0}^{k-1} S_i \)

\[
P_n(t) = P\{W_n \leq t < W_{n+1}\}
\]

\[
P\{S_0 \leq t\} = P\{W_1 \leq t\} = 1 - P\{X(t) = 0\} = 1 - P_0(t) = 1 - e^{-\lambda_0 t}
\]

Solution of differential equations

Introduce \( Q_n(t) = e^{\lambda_0 t} P_n(t) \), then

\[
Q_n(t) = \lambda_n e^{\lambda_0 t} P_n(t) + e^{\lambda_0 t} P'_n(t) = e^{\lambda_0 t} (\lambda_n P_n(t) + P'_n(t)) = e^{\lambda_0 t} \lambda_n P_{n-1}(t)
\]

such that

\[
Q_n(t) = \lambda_{n-1} \int_0^t e^{\lambda_0 x} P_{n-1}(x) \, dx
\]

leading to

\[
P_n(t) = \lambda_{n-1} e^{-\lambda_0 t} \int_0^t e^{\lambda_0 x} P_{n-1}(x) \, dx
\]
Regular Process

\[ \sum_{n=0}^{\infty} P_n(t) = 1 \]

True if:

\[ \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{\lambda_k} = \infty \]

Then

\[ \sum_{k=0}^{\infty} P_k(t) = 1 \]

Recursive full solution when \( \lambda_i \neq \lambda_j \) for \( i \neq j \)

\[ P_n(t) = \left( \prod_{j=0}^{n-1} \lambda_j \right) \sum_{j=0}^{n} B_{i,n} e^{-\lambda_j t} \]

with

\[ B_{i,n} = \prod_{j \neq i} (\lambda_j - \lambda_i)^{-1} \]

Yule Process

\[ P'_n(t) = -\beta n P_n(t) + \beta (n-1) P_{n-1}(t) \]

\[ P_n(t) = e^{-\beta t} \left( 1 - e^{-\beta t} \right)^{n-1} \]

Death Process Postulates

1. \( P\{X(t+h) = k-1|X(t) = k\} = \mu_k h + o(h) \)
2. \( P\{X(t+h) = k|X(t) = k\} = 1 - \mu_k h + o(h) \)
3. \( X(0) = N \)

\[ P_n(t) = \left( \prod_{j=0}^{n-1} \mu_j \right) \sum_{j=n}^{N} A_{j,n} e^{-\lambda_j t} \]

with

\[ A_{k,n} = \prod_{j=n,j \neq k}^{N} (\mu_j - \mu_k)^{-1} \]

For \( \mu_k = k \mu \) we have by a simple probabilistic argument

\[ P_n(t) = \binom{N}{n} (e^{-\mu t})^n (1 - e^{-\mu t})^{N-n} = \binom{N}{n} e^{-n \mu t} (1 - e^{-\mu t})^{N-n} \]

Birth and Death Process Postulates

1. \( P_{i,i+1}(h) = \lambda_i h + o(h) \)
2. \( P_{i,i-1}(h) = \mu_i h + o(h) \)
3. \( P_{i,i}(h) = -(\lambda_i + \mu_i)h + o(h) \)
4. \( P_{i,j}(0) = \delta_{ij} \)
5. \( \mu_0 = 0, \lambda_0 > 0, \mu_i, \lambda_j > 0, i = 1, 2, \ldots \)
Infinitesimal Generator

\[
A = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & 0 \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 \\
0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \\
0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[
P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s), \quad P(t+s) = P(t)P(s)
\]

Regular Process

\[
\sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{k=0}^{n} \theta_k = \infty
\]

where

\[
\theta_0 = 1, \quad \theta_n = \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}
\]

ODE's for Birth and Death Process

\[
P_{0j}'(t) = -\lambda_0 P_{0j}(t) + \lambda_1 P_{1j}(t)
\]

\[
P_{ij}'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t) + \lambda_{i+1}P_{i+1,j}(t)
\]

\[
P_{ij}(0) = \delta_{ij}
\]

\[
P'(t) = AP(t)
\]

Backward Kolomogorov equations

\[
P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t)
\]

\[
= P_{i-1,j}(t)P_{i,j}(h) + P_{i,j}(h)P_{i+1,j}(t) + P_{i+1,j}(h)P_{i,j}(t) + o(h)
\]

\[
= \mu_i h P_{i-1,j}(t) + (1 - (\mu_i + \lambda_i)h)P_{i,j}(t) + \lambda_i h P_{i+1,j}(t) + o(h)
\]

Forward Kolomogorov equations

\[
P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(h)
\]

\[
P'(t) = P(t)A
\]

The backward and forward equations have the same solutions in all "ordinary" models, that is models without explosion and models without instantaneous states.
ODE's for Birth and Death Process

\[ P'_{i0}(t) = -P_{i0}(t)\lambda_0 + P_{i1}(t)\mu_1 \]
\[ P'_{ij}(t) = P_{i,j-1}\lambda_{j-1} - P_{j\bar{j}}(t)\lambda_j + \mu_j + P_{ij+1}(t)\mu_{j+1} \]
\[ P'_{i0}(0) = \delta_{\bar{i}} \]
\[ P'(t) = AP \]

Sojourn times

\[ P\{S_i \geq t\} = G_i(t) \]
\[ G_i(t+h) = G_i(t)G_i(h) = G_i(t)[P_{i(h)} + o(h)] \]
\[ G_i(t)[1 - (\lambda_i + \mu_i)h] + o(h) \]
\[ G'_i(t) = -(\lambda_i + \mu_i)G_i(t) \]
\[ G_i(t) = e^{-(\lambda_i+\mu_i)t} \]

Embedded Markov chain

Define \( T_n \) as the time of the \( n \)th state change at the Define \( N(t) \) to be number of state changes up to time \( t \).

\[ P\{X(T_{n+1}) = j | X(T_n) = i\} \]

Define \( Y_n = X(T_n) \)

\[ P\{Y_{n+1} = j | Y_n = i\} = \begin{cases} 0 & \text{for } j = i - 1 \\ \frac{\mu_i}{\mu_i + \lambda_i} & \text{for } j = i + 1 \\ 0 & \text{for } j \notin \{i - 1, i + 1\} \end{cases} \]

\[ P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{\mu_1}{\mu_1 + \lambda_1} & 0 & \frac{\lambda_1}{\mu_1 + \lambda_1} & 0 & \cdots \\ 0 & \frac{\mu_2}{\mu_2 + \lambda_2} & 0 & \frac{\lambda_2}{\mu_2 + \lambda_2} & \cdots \\ 0 & 0 & \frac{\mu_3}{\mu_3 + \lambda_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

Definition through Sojourn Times and Embedded Markov Chain

Sequence of states governed by the discrete Time Markov chain with transition probability matrix \( P \)

Exponential sojourn times in each state with intensity parameter \( \gamma_i(= \mu_1 + \lambda_i) \)
Linear Growth with Immigration

\[ P_{i0}(t) = -aP_{i0}(t) + \mu P_{i1}(t) \]
\[ P_{ij}(t) = [\lambda(j - 1) + a]P_{ij-1}(t) + \mu(j + 1)P_{i,j+1}(t) \]

With \( M(0) = i \) if \( X(0) \) this leads to

\[ \mathbb{E}[X(t)] = M(t) = \sum_{j=1}^{\infty} jP_{ij}(t) \]
\[ M'(t) = a + (\lambda - \mu)M(t) \]
\[ M(t) = \left\{ \begin{array}{ll}
\frac{at + i}{\lambda - \mu} & \text{if } \lambda = \mu \\
\frac{at + i}{\lambda - \mu} \{ e^{(\lambda - \mu)t} - 1 \} + ie^{(\lambda - \mu)t} & \text{if } \lambda \neq \mu
\end{array} \right. \]

Two-State Markov Chain

\[ \mathbf{A} = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \]
\[ P'_{00}(t) = -\alpha P_{00}(t) + \beta P_{01}(t) \]

With \( P_{01}(t) = 1 - P_{00}(t) \) we get
\[ P'_{00}(t) = -(\alpha + \beta)P_{00}(t) + \beta \]

Using the standard approach with \( Q_{00}(t) = e^{(\alpha + \beta)t}P_{00}(t) \) we get
\[ Q_{00}(t) = \frac{\beta}{\alpha + \beta} e^{(\alpha + \beta)t} + C \]

which with \( P_{00}(0) = 1 \) give us
\[ P_{00}(t) = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} e^{-(\alpha + \beta)t} = \pi_1 + \pi_2 e^{-(\alpha + \beta)t} \]

with \( \pi = (\pi_1, \pi_2) = \left( \frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right) \).

Two-State Markov Chain - continued

Using \( P_{01}(t) = 1 - P_{00}(t) \) we get
\[ P_{01}(t) = \pi_2 - \pi_2 e^{-(\alpha + \beta)t} \]

and by an identical derivation
\[ P_{11}(t) = \pi_2 + \pi_1 e^{-(\alpha + \beta)t} \]
\[ P_{10}(t) = \pi_1 - \pi_1 e^{-(\alpha + \beta)t} \]