Lecture 4: Discrete-time Markov Chains II

The period of a state

Let the chain start in state \( i \).
Consider the times where a revisit is possible
\[
\{ n : p_{ii}(n) > 0 \}
\]
The greatest common divisor of these times are the period of state \( i \).
Examples: In a simple (a)symmetric random walk, all states are periodic with period 2.
In the growth/rest process, all states are aperiodic.

Transience vs. persistency/recurrency

As before, let \( N_i \) be the number of visits to state \( i \), i.e.
\[
N_i = \# \{ n \geq 1 : X_n = i \}.
\]
Likewise, let \( T_i \) be the time of first (re)visit to state \( i \), i.e.
\[
T_i = \min \{ n \geq 1 : X_n = i \}.
\]

<table>
<thead>
<tr>
<th>State</th>
<th>Definition</th>
<th>Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transient</td>
<td>( P^i(N_i &gt; 0) &lt; 1 )</td>
<td>( \sum_{n \geq 1} p_{ii}(n) &lt; \infty )</td>
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<tr>
<td>Null persistent</td>
<td>( P^i(N_i &gt; 0) = 1 )</td>
<td>( \sum_{n \geq 1} p_{ii}(n) = \infty )</td>
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<td></td>
<td>( E^iT_i = \infty )</td>
<td>( \limsup_{n \to \infty} p_{ii}(n) = 0 )</td>
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<tr>
<td>Positive recurrent</td>
<td>( P^i(N_i &gt; 0) = 1 )</td>
<td>( \sum_{n \geq 1} p_{ii}(n) = \infty )</td>
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<td>( E^iT_i &lt; \infty )</td>
<td>( \limsup_{n \to \infty} p_{ii}(n) &gt; 0 )</td>
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</table>

For a transient state \( i \): \( N_i \) is geometrically distributed and \( T_i \) is infinite with non-zero probability (under \( P^i \)).
For a persistent state, \( T_i \) is finite w.p. 1, and \( N_i \) is infinite w.p. 1 (under \( P^i \)).
State classification in general random walk

\[ S_{n+1} = S_n + X_n, \quad S_0 = 0 \]

where \( X_n \) are i.i.d. with mean \( \mu \) and variance \( \sigma^2 \).

According to the Central Limit Theorem

\[ S_n \sim N(n\mu, n\sigma^2) \quad \text{so} \quad p_{00}(n) \approx \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-\frac{n\mu^2}{2\sigma^2}} \]

Corollary (4) (p. 221) says that the state 0 is persistent iff \( \sum_n p_{00}(n) = \infty \). I.e., iff \( \mu = 0 \).

Compare exercise 6.3.2 where we had \( X_n \in \{-1, 2\} \).

Random walks in higher dimensions

A vector process in \( N \) dimensions \( X_n = (X_n^{(1)}, \ldots, X_n^{(N)}) \), where the co-ordinate processes \( X_n^{(i)} \) are independent biased random walks with mean \( \mu_i \) and variance \( \sigma_i^2 \).

\[ p_{00}(n) \approx \frac{1}{(2\pi n)^{N/2} \prod \sigma_i} e^{-\frac{n}{2} \sum \frac{\mu_i^2}{\sigma_i^2}} \]

The origin is null persistent iff \( \forall i : \mu_i = 0 \) and \( n \leq 2 \); otherwise transient.

(The conclusion can be generalized to the situation where the co-ordinate processes are correlated)

Connecting random walks and diffusion

Compare the diffusion equation in \( C(x, t) \):

\[ \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \]

with a symmetric random walk \( S_n \) with \( p_{i(i+1)} = p_{i(i-1)} = p, \quad p_{ii} = 1 - 2p \).

1. When \( n \) is large, \( S_n \) is approximately distributed as \( N(0, \sigma^2 n) \).

   Compare this with the solution of the diffusion equation with initial condition \( C(x, 0) = \delta(x) \)

   \[ C(x, t) = \frac{1}{\sqrt{2\pi 2Dt}} e^{-\frac{x^2}{22Dt}} \]

2. Compare the forward equation for the random walk with the second-order central finite difference scheme for discretisation of the diffusion equation. (If you have studied numerics ...)

Survival analysis with competing hazards

A model of the course of university studies of a single student:

\[ X_n \in \mathbb{N} \cup \{G\} \cup \{A\} \]

where \( X_n = i \) means that the student is at his/her \( i \)th semester,
\( X_n = G \) means graduated,
\( X_n = A \) means studies abandoned without graduation.

Transition probabilities:

\[ p_{iG} = g_i, \quad p_{iA} = a_i, \quad p_{i(i+1)} = 1 - g_i - a_i \]

We can divide the state space into three: The transient states \( \mathbb{N} \), the absorbing state \( G \) and the absorbing state \( A \).
Classification of state space

Communicating states

State \( i \) communicates with \( j \) (written \( i \rightarrow j \)), if \( p_{ij}(m) > 0 \) for some \( m \).

In terms of the graph: If there is a path from \( i \) to \( j \).

\( i \) and \( j \) intercommunicates (written \( i \leftrightarrow j \)) if \( i \rightarrow j \) and \( j \rightarrow i \). (If there is a closed path containing both \( i \) and \( j \)).

\( \leftrightarrow \) is an equivalence relation on state space \( S \).

Two intercommunicating states must have the same qualitative properties (theorem 2): Same period, same persistency.

Closed and irreducible subsets of state space

A closed set \( C \) is one which can never be departed, once entered: \( p_{ij} = 0 \) for all \( i \in C \) and all \( j \not\in C \).

An irreducible set \( C \) is one in which all states intercommunicate.

The Decomposition theorem says that state space can be partitioned into the transient states \( T \) and closed irreducible sets \( C_i \):

\[ S = T \cup C_1 \cup C_2 \cup \cdots \]

When studying long-time behaviour, we can concentrate on the cases \( S = T \) and \( S = C_1 \).
Stationary distributions

We had

\[ \mu_{n+1} = \mu_n P \]

A distribution \( \pi \) on \( S \) (such that \( \pi_j \geq 0 \) and \( \sum_j \pi_j = 1 \)) is stationary iff

\[ \pi = \pi P \]

This is important: In many applications, we only care about stationary distributions.

The symmetric simple random walk

The stationarity equation \( \pi = \pi P \) reads

\[ \pi_j = \frac{1}{2} (\pi_{j-1} + \pi_{j+1}) \]

on the interior. The general solution is

\[ \pi_j = a j + b \]

1. With two reflecting barriers, \( \pi_j = b \) on the interior.
2. With no barriers, no distribution can live up to this. So what can we do with the solution \( \pi_j = 1 \)?

A single server handles customers one at a time. There are two types of events: New customers arriving, and the completion of service of a customer.

\( S_n \) models the queue length immediately after event no. \( n \).

When the queue is non-empty, the next event is either a new customer arriving (with probability \( p \)) or a customer departing (w.p. \( q = 1 - p \)).

Result: A stationary queue length distribution exists iff the mean service time is smaller than the mean interarrival time.

If the two times are equal, then \( S_n \) is null recurrent.

If the mean service time is the greater, then all states are transient. The queue length grows to infinity.

The asymmetric simple random walk

The stationarity equation \( \pi = \pi P \) reads

\[ \pi_j = \pi_{j-1} p + \pi_{j+1} q \]

on the interior. The general solution is

\[ \pi_j = c \cdot \left( \frac{p}{q} \right)^j + k \]

With \( p < \frac{1}{2} \) and a lower reflecting barrier at 0, we find \( k = 0 \), and a geometrically decaying stationary distribution \( \pi \).
Theorem 3: Existence of stationary distributions

An irreducible chain has a stationary distribution $\pi$ iff all states are positive recurrent. In this case, $\pi$ is unique and is given by

$$\pi_i = \frac{1}{\mu_i}$$

where $\mu_i$ is the mean recurrence time of $i$.

**Theorem 6:**

If the chain is irreducible and persistent, there exists a unique (up to a multiplicative constant) positive root $x$ of $x = xP$. The chain is non-null iff $\sum x_i < \infty$.

**Examples:**

1. The unrestricted simple random walk: $x_i = 1$ for all $i$. The chain is null persistent.
2. Any irreducible chain on a finite state space is non-null.

A criterion for transience

According to theorem 3, if we can find a stationary distribution, then the chain is positive recurrent.

**Theorem (10):** Fix an arbitrary target state $s \in S$. The chain is transient iff there exists a bounded non-zero solution to

$$y_i = \sum_{j:j \neq s} p_{ij}y_j, \quad i \neq s.$$ 

The candidate solution is

$$y_i = \mathbb{P}(\text{the chain never visits state } s | X_0 = i)$$

To verify this solution, condition on first transition.

Limiting distributions

**Theorem (17):** For an irreducible aperiodic chain

$$p_{ij}(n) \to \frac{1}{\mu_j}$$

as $n \to \infty$.

In the transient or null persistent case, $\mu_j = \infty$ so $p_{ij}(n) \to 0$.

In the non-null persistent case, $p_{ij}(n) \to \pi_j$, the stationary distribution.

Note: The limit probability does not depend on initial state.
Summary

- Partitioning of state space
  ... allows us to assume irreducibility
- We have defined stationary distributions
  ... shown that the transition probabilities converge to the stationary distribution in the long time
  ... related the stationary distribution to mean recurrence times
  ... and given criteria for when the stationary distribution exists

Next week

Section 6.5 on time reversible processes
Section 6.6 on finite state spaces.
Section 6.8 on Birth and Poisson processes (if we have time).

Exercises

Problem 3.12 from Wolff.
Exercise uht-01.
Exercise 6.4.8 from G & S. Hint: Do not find the stationary distribution by solving the equation \( \pi = \pi P \). Rather use the properties of Poisson distributions.
Exercise 6.3.1 from G & S. Hint: Use material from section 6.4 rather than 6.3. Assume \( r < 1, a_0 < 1 \). Start by focusing on the state 0.