The Poisson Process

Definition (in terms of Poisson distribution)

The law of rare events

Definition in terms of intervals

Uniformity

Spatial, compound, and marked Poisson processes

Next week

Birth processes

Death processes

Birth and death processes

Two weeks from now

Limiting behaviour of birth and death processes

Birth and death processes with absorption

Finite continuous time Markov chains

The Poisson process

Definition

A Poisson process, or rate, \( p \), is an integer-indexed integer-valued stochastic process \( \{X(t); t \geq 0\} \) for which

1. For any time points \( t_0 = 0 < t_1 < t_2 < \cdots < t_n \), the process increments

\[
X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})
\]

are independent random variables;

2. For \( s \geq 0 \) and \( t \geq 0 \), the random variable \( X(t + s) - X(s) \) has the binomial distribution

\[
\Pr \{X(t + s) - X(s) = k\} = \binom{t}{k} p^k (1 - p)^{t-k}
\]

3. \( X(0) = 0 \).

The Bernoulli Process

Definition

A Bernoulli process of parameter, or rate, \( p \), is an integer-indexed integer-valued stochastic process \( \{X(t); t \geq 0\} \) for which

- The next fundamental process
- As important as discrete time Markov chain
- Continuous time model
- Parallel to Bernoulli process
- Model for complete randomness
- Three different characterisations
Bernoulli Process Waiting times and intensities

Define $W_n$ waiting time to the $n$th event

1. The waiting time $(W_1)$ to the first event (and the waiting time $W_{n+1} - W_n$ between the $n$ and the $(n+1)$st event) is geometric, $\mathbb{P}\{W_1 = k\} = p(1-p)^{k-1}, \mathbb{P}\{W_1 > k\} = (1-p)^k, k = 1, 2, 3, \ldots.$

2. The waiting time to the $n$th event follows a negative binomial distribution $\mathbb{P}\{W_n = k\} = \binom{k-1}{n-1}p^n(1-p)^{k-n},$ for $k = n, n+1, \ldots$.

The Probability (intensity) of having an event in a single interval is

1. $\mathbb{P}\{X(t+1) - X(t) = 0\} = 1 - p$
2. $\mathbb{P}\{X(t+1) - X(t) = 1\} = p$
3. $\mathbb{P}\{X(t+1) - X(t) > 1\} = 0$

Bo Friis Nielsen  Poisson Processes

The Poisson Process

Definition (Page 225)
A Poisson process of intensity, or rate, $\lambda > 0$, is an integer-valued stochastic process $\{X(t); t \geq 0\}$ for which

1. For any time points $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, the process increments
   
   $X(t_i) - X(t_{i-1})$ are independent random variables;

2. For $s \geq 0$ and $t \geq 0$, the random variable $X(t+s) - X(s)$ has the Poisson distribution

   $\mathbb{P}\{X(t+s) - X(s) = k\} = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$, for $k = 0, 1, \ldots$

3. $X(0) = 0.$

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Poisson process - three different characterisations

- Independence assumptions in all cases
  - Number of events in an interval is Poisson distributed with independent increments
  - Constant intensity for event
  - Intervals between event are exponentially distributed

Intensity characterisation - infinitessimal probabilities

- $\mathbb{P}\{\text{one event in an interval of length } \Delta t\}$
  $\mathbb{P}\{X(t + \Delta t) - X(t) = 1\} = \Delta t \lambda + o(\Delta t)$ Where
  
  $\frac{o(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$

- $o(t)$ is a function that tends to zero faster than $t$

- $\mathbb{P}\{\text{no event in an interval of length } \Delta t\}$
  $\mathbb{P}\{X(t + \Delta t) - X(t) = 0\} = 1 - \Delta t \lambda + o(\Delta t)$

- $\mathbb{P}\{\text{more than one event during } \Delta t\} = o(\Delta t)$
Nonhomogeneous Poisson Process

\[ P\{X(t+h) - X(t) = 1\} = \frac{(\lambda h) e^{-\lambda h}}{h!} \]
\[ = (\lambda h) \left( 1 - \lambda h + \frac{1}{2} \lambda^2 h^2 - \cdots \right) \]
\[ = \lambda h + o(h) \]

where \( \lim_{h \to 0} \frac{o(h)}{h} = 0 \). If we assume

\[ P\{X(t+h) - X(t) = 1\} = \lambda t(h) + o(h) \]

\[ P\{X(t+h) - X(t) = 0\} = 1 - \lambda t(h) + o(h) \]

\[ X(t) - X(s) \sim \text{Pois} \left( \int_s^t \lambda(u)du \right) \]

Homogeneity transformation \( Y(s) = X(t) \) with \( s = \int_0^t \lambda(u)du \)

The Law of Rare Events

**Theorem (5.3 Page 233)**

Let \( \epsilon_1, \epsilon_2, \ldots \) be independent Bernoulli variables, where

\[ P\{\epsilon_i = 1\} = p_i \quad \text{and} \quad P\{\epsilon_i = 1\} = 1 - p_i \]

and let \( S_n = \epsilon_1 + \cdots + \epsilon_n \). The exact probabilities for \( S_n \) are given by

\[ P\{S_n = k\} = \sum_{i=1}^{(k)} \prod_{j=1}^{i} p_i^{x_i} (1 - p_i)^{1-x_i}, \]

where \( \sum_{i=1}^{(k)} \) denotes the sum over all 0,1 valued \( x_i \)'s such that

\( x_1 + \cdots + x_n = k \), and Poisson probabilities with

\( \mu = p_1 + \cdots + p_n \) differ at most by

\[ \left| P\{S_n = k\} - \frac{\mu^k}{k!} e^{-\mu} \right| \leq \sum_{i=1}^{n} p_i^2 \]

From Poisson distribution to exponential distribution

\( X(t) \in P(\lambda t) \)

\[ P\{X(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \]

\[ P\{S_0 > t\} = P\{X(t) = 0\} = e^{-\lambda t} \]

\[ P\{S_0 \leq t\} = F(t) = 1 - e^{-\lambda t} \]

\( W_n \) time of the \( n \)th event

\( S_n \) time between the \( n \)th and \( n + 1 \)st event (sojourn time).

The time between two consecutive events is exponentially distributed

\[ S_n \in \text{exp}(\lambda) \quad P\{S_n \leq t\} = F(t) = 1 - e^{-\lambda t} \]

The intervals are iid.

Important relation between \( S_n \) and \( X(t) \)

\[ P\{W_n \leq t\} = P\{X(t) \geq n\} \]
Time to the $n$th event - the Erlang distribution

- The relation between $W_n$ and $X(t)$

\[ P\{W_n \leq t\} = P\{X(t) \geq n\} \]

\[ P\{W_n \leq t\} = P\{X(t) \geq n\} = \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \]

- $E(S_n) = \frac{1}{\lambda}$, $E(W_n) = \frac{n}{\lambda}$
- $\text{Var}(S_n) = \frac{1}{\lambda^2}$, $\text{Var}(W_n) = \frac{n}{\lambda^2}$
- We say that $W_n \in \text{Erl}_n(\lambda)$
- The Erlang distribution can be interpreted as the distribution for the sum of independent exponential random variables.

Uniform Distribution and the Poisson Process

**Theorem (5.7 Page 248)**

Let $W_1, W_2, \ldots$ be the occurrence times in a Poisson process $X(t)$ of rate $\lambda > 0$. Conditioned on $X(t) = n$ the random variables $W_1, W_2, \ldots$ have the joint probability density function

\[ f_{W_1, \ldots, W_n|X(t)=n}(w_1, \ldots, w_n) = n! t^n e^{-\beta w_1 - \lambda (w_1 + \cdots + w_n)} \]

\[ M = \mathbb{E}\left[ \sum_{k=1}^{X(t)} e^{-\beta W_k} \right] \]

\[ M = \sum_{n=1}^{\infty} \mathbb{E}\left[ \sum_{k=1}^{n} e^{-\beta W_k} \Big| X(t) = n \right] \mathbb{P}\{X(t) = n\} \]

\[ \mathbb{E}\left[ \sum_{k=1}^{X(t)} e^{-\beta W_k} \big| X(t) = n \right] = \mathbb{E}\left[ \sum_{k=1}^{n} e^{-\beta U_k} \right] \]

**Shot Noise**

\[ I(t) = \sum_{k=1}^{X(t)} h(t - W_k) \]

\[ \mathbb{E}(I(t)) = \lambda \int_0^t h(u) du, \quad \text{Var}(I(t)) = \lambda \int_0^t h(u)^2 du \]
Spatial Poisson Processes

1. For each \( A \) in \( \mathcal{A} \), the random variable \( N(A) \) has a Poisson distribution with parameter \( \lambda |A| \).
2. For every finite collection \( \{A_1, \ldots, A_n\} \) of disjoint subsets of \( S \), the random variables \( N(A_1), \ldots, N(A_n) \) are independent.

3. The possible values for \( N(A) \) are the nonnegative integers \( \{0, 1, 2, \ldots\} \) and \( 0 < \Pr\{N(A) = 0\} < 1 \) if \( 0 < |A| < \infty \).

4. The probability distribution of \( N(A) \) depends on the set \( A \) only through its size (length, area, volume) \( |A| \), with the further property that \( \Pr\{N(A) \geq 1\} = \lambda |A| + o(|A|) \) as \( |A| \to 0 \).

5. For \( m = 2, 3, \ldots \), if \( A_1, A_2, \ldots, A_m \) are disjoint regions, then \( N(A_1), N(A_2), \ldots, N(A_m) \) are independent random variables and \( N(A_1 \cup A_2 \cup \cdots \cup A_m) = N(A_1) + N(A_2) + \cdots + N(A_m) \).

6. \( \lim_{|A| \to 0} \frac{\Pr\{N(A) \geq 1\}}{\Pr\{N(A) = 1\}} = 1 \).

Compound (Reward) Poisson Processes

We have random variables \( Y_1, Y_2, \ldots \) with cumulative distribution function

\[ G(y) = \Pr\{Y \leq y\}, \quad \mathbb{E}(Y_i) = \mu, \quad \text{Var}(Y_i) = \nu^2 \]

A Compound Poisson Process (reward process) is defined by

\[ Z(t) = \sum_{k=1}^{X(t)} Y_k \]

\[ \mathbb{E}[Z(t)] = \mu t, \quad \text{Var}[Z(t)] = \lambda t(\mu^2 + \nu^2) \]

Conditional uniform distribution

\[ \Pr\{N(B) = 1 \mid N(A) = 1\} = \frac{|B|}{|A|} \text{ for any set } B \subset A \]

For \( A_1 \cup A_2 \cup \cdots A_m = A \)

\[ \Pr\{N(A_1) = k_1, \ldots, N(A_m) = k_m \mid N(A) = n\} = \frac{n!}{k_1! \cdots k_m!} \left( \frac{|A_1|}{|A|} \right)^{k_1} \cdots \left( \frac{|A_m|}{|A|} \right)^{k_m} \]

\[ \mathbb{P}\{Y_1 + \cdots + Y_n \leq y\} = \int_{-\infty}^{\infty} G^{n-1}(y-z) \, dG(z) \]

\[ \mathbb{P}\{Z(t) \leq z\} = \mathbb{P}\left\{ \sum_{k=1}^{X(t)} Y_k \leq z \right\} \]

\[ = \sum_{n=0}^{\infty} \mathbb{P}\left\{ \sum_{k=1}^{X(t)} Y_k \leq z \mid X(t) = n \right\} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \]

\[ = \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} G^{(n)}(z) \]
Let $T$ be the time to get beyond critical level $a$

\[ \{ T > t \} \text{ if and only if } \{ Z(t) \leq a \} \]

\[
E[T] = \int_0^\infty P\{ T > t \} \, dt
\]

\[
= \sum_{n=0}^\infty \left( \int_0^\infty (\lambda t)^n e^{-\lambda t} \, \frac{t^n}{n!} \, dt \right) G^{(n)}(a)
\]

\[
= \lambda^{-1} \sum_{n=0}^\infty G^{(n)}(a)
\]

Additional Reading

Erhan Çinlar: “Introduction to Stochastic Processes”