Stationary Ornstein-Uhlenbeck process

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We define a bivariate normal vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right).$$

With

$$\boldsymbol{x} \quad = \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

we can express the joint density as

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sqrt{\det(\boldsymbol{\Sigma})}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}} \end{aligned}$$

we have the conditional density

$$f_{X_2|X_1=x_1}(x_2) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}} = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{-\frac{(x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1))^2}{2\sigma_2^2(1-\rho^2)}}$$

to get

$$\mathbb{E}(X_2|X_1 = x_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \quad \mathbb{E}(X_2|X_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X_1 - \mu_1)$$

or

$$\mathbb{E}(X_2 - \mu_2|X_1 = x_1) = \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), \quad \mathbb{E}(X_2 - \mu_2|X_1) = \rho\frac{\sigma_2}{\sigma_1}(X_1 - \mu_1).$$

To ease notation and with no real loss of generality we assume $\mu_1 = \mu_2 = 0$

$$\mathbb{E}(X_2|X_1 = x_1) = \rho\frac{\sigma_2}{\sigma_1}x_1, \quad \mathbb{E}(X_2 - \mu_2|X_1) = \rho\frac{\sigma_2}{\sigma_1}X_1$$

and we rewrite to get

$$\sigma_1^2 \mathbb{E}\left(X_2 | X_1 = x_1\right) = \rho\sigma_1\sigma_2 x_1, \quad \sigma_1^2 \mathbb{E}\left(X_2 | X_1\right) = \rho\sigma_1\sigma_2 X_1$$

to finally obtain

$$\sigma_1^2 \mathbb{E}\left(X_2 | X_1\right) = \mathbb{C}\mathrm{ov}\left(X_1, X_2\right) X_1.$$

We now want to construct a one-dimensional Gaussian process where this relation holds for any pair $(X(t_1), X(t_2))$, so we assume that the covariance function $\Gamma(t_1, t_2) = \mathbb{C}\mathrm{ov}((X(t_1), X(t_2))$ is time homogenous such that $\mathbb{C}\mathrm{ov}((X(t_1), X(t_2)) = \Gamma(t_2 - t_1)$. Our assumption amounts to $\sigma^2 \mathbb{E}\left(X(t) | X(0)\right) = \Gamma(t) X(0)$ or $\Gamma(0) \mathbb{E}\left(X(t) | X(0)\right) = \Gamma(t) X(0)$ We now evaluate $\Gamma(t_1 + t_2)$.

$$\begin{aligned}
\Gamma(t_1 + t_2) &= \mathbb{E}[X(0) X(t_1 + t_2)] = \mathbb{E}\left[\mathbb{E}(X(0) X(t_1 + t_2) | X(0), X(t_1))\right] \\
&= \mathbb{E}\left[X(0) \mathbb{E}(X(t_1 + t_2) | X(0), X(t_1))\right] = \mathbb{E}\left[X(0) \mathbb{E}(X(t_1 + t_2) | X(t_1))\right] = \mathbb{E}\left[X(0) \frac{1}{\sigma^2} \Gamma(t_2) X(t_1)\right] \\
&= \frac{1}{\sigma^2} \Gamma(t_1) \Gamma(t_2),
\end{aligned}$$

and get the functional equation

$$\sigma^2 \Gamma(t_1 + t_2) = \Gamma(t_1) \Gamma(t_2),$$

with solution

$$\Gamma(t) = \sigma^2 e^{-\alpha |t|}$$

for some α. We have defined a stationary Gaussian process with $X(t) \sim \mathrm{N}(0, \sigma^2)$, i.e. $\mu(t) = 0$ and covariance function $\Gamma(t) = \sigma^2 e^{-\alpha |t|}$. This process is called the stationary Ornstein-Uhlenbeck process.

The approach is taken from [1] Section 9.6 Page 407.

References

[1] G. R. Grimmett and D. R. Stirzacker. *Probability and Random Processes.* Oxford University Press, third edition, 1995.