Moment generating function and Laplace transform

02407 Stochastic Processes
2018-11-5
BFN/bfn

We have used the probability generating function a couple of times. The most general transform is the characteristic function, which is discussed in some depth in Chapter 11 of the textbook. This note summarises results for two other commonly used transform. The transforms are all of the same family primarily with slight variations in the formulation in terms of the argument.

Moment generating function

We define the moment generating function to be:

$$H(\theta) = \mathbb{E}\left(e^{\theta X}\right) = \int_{-\infty}^{\infty} e^{\theta x} \mathrm{d}F(x)$$

for all θ where the integral converges.

Laplace transform

Similarly, usually only for non-negative random variables, we define the Laplace transform to be:

$$L(\theta) = \mathbb{E}\left(e^{-\theta X}\right) = \int_{0}^{\infty} e^{-\theta x} \mathrm{d}F(x)$$

for all θ where the integral converges.

Finite sums

For X_1 and X_2 independent we have

$$H_{X_1+X_2}(\theta) = \mathbb{E}\left(e^{\theta(X_1+X_2)}\right) = \mathbb{E}\left(e^{\theta X_1} e^{\theta X_2}\right) = \mathbb{E}\left(e^{\theta X_1}\right) \mathbb{E}\left(e^{\theta X_2}\right) = H_{X_1}(\theta) H_{X_2}(\theta).$$

and $L_{X_1+X_2}(\theta) = L_{X_1}(\theta) L_{X_2}(\theta)$.

Moments

$$\mathbb{E}\left(X^n\right) = \left.\frac{\mathrm{d}^n H(\theta)}{\mathrm{d}^n \theta}\right|_{\theta=0} = (-1)^n \left.\frac{\mathrm{d}^n L(\theta)}{\mathrm{d}^n \theta}\right|_{\theta=0},$$

whenever $\mathbb{E}\left(X^n\right)$ is finite.

The renewal function $M(x)$ is

$$M(x) = \sum_{n=1}^{\infty} F_n(x)$$

where $F_n(x) = \int_0^x F_{n-1}(x-u)\mathrm{d}F(x)$ i.e. the n'th convolution of F with itself and thus the distribution of $W_n = \sum_{k=1}^n X_k$. Assuming a density and taking transforms we get

$$\begin{aligned}
\int_0^\infty e^{-\theta x} M(x)\mathrm{d}x &= \int_0^\infty e^{-\theta x} \sum_{n=1}^\infty F_n(x)\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty e^{-\theta x} F_n(x)\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty e^{-\theta x} \int_0^x f_n(u)\mathrm{d}u\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_0^\infty \int_u^\infty e^{-\theta x} f_n(u)\mathrm{d}x\mathrm{d}u = \theta^{-1} \sum_{n=1}^\infty \int_0^\infty e^{-\theta u} f_n(u)\mathrm{d}u = \theta^{-1} \sum_{n=1}^\infty L_n(\theta) \\
&= \theta^{-1} \sum_{n=1}^\infty L(\theta)^n = \frac{L(\theta)}{\theta(1 - L(\theta))}
\end{aligned}$$

We can handle the general case too with a slight reformulation

$$\begin{aligned}
\int_0^\infty e^{-\theta x} M(x)\mathrm{d}x &= \int_0^\infty e^{-\theta x} \sum_{n=1}^\infty F_n(x)\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty e^{-\theta x} F_n(x)\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty e^{-\theta x} \int_0^x \mathrm{d}(F_n(u))\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_0^\infty \int_u^\infty e^{-\theta x} \mathrm{d}x\mathrm{d}(F_n(u)) = \theta^{-1} \sum_{n=1}^\infty \int_0^\infty e^{-\theta u} \mathrm{d}(F_n(u)) = \theta^{-1} \sum_{n=1}^\infty L_n(\theta) \\
&= \theta^{-1} \sum_{n=1}^\infty L(\theta)^n = \frac{L(\theta)}{\theta(1 - L(\theta))}
\end{aligned}$$