Notes from l.11 Oct. 7. Reversible Markov Chains

We consider a discrete time Markov chain with transition probability matrix $\underset{\sim}{P}$ and inititial probability vector $\vec{a}$. Define by $q_{ij}^{(n+1)} = Prob(X_n = j|X_{n+1} = i)$. It is easy to show that the sequence $X_{n+1}, X_n, X_{n-1}, \ldots$ is itself a discrete time Markov chain.

$$q_{ij}^{(n+1)} = Prob(X_n = j|X_{n+1} = i) = \frac{Prob(X_n = j, X_{n+1} = i)}{Prob(X_{n+1} = i)} =$$

$$\frac{Prob(X_{n+1} = i|X_n = j)Prob(X_n = j)}{Prob(X_{n+1} = i)} = \frac{p_{ji}a_j^{(n)}}{a_i^{(n+1)}}$$

We will denote the new sequence the *reversed* chain. The transition probabilities is not stationary in general. For a stationary Markov chain, however, we have $\vec{a} = \vec{\pi}$ and thus

$$q_{ij} == \frac{p_{ji}\vec{\pi}_j^{(n)}}{\vec{\pi}_i^{(n+1)}}$$

By denoting the diagonal matrix with $\vec{\pi}$ on the diagonal by Δ we can derive the matrix expression $\mathbf{Q}$=$\Delta^{-1}\mathbf{P}\Delta$.

A Markov chain will be called *reversible* if it is stationary equivalent with its reversed, ie. $\mathbf{Q}$=$\mathbf{P}$. Elementwise this corresponds to $\pi_j p_{ji} = \pi_i p_{ij}$. That is the expected number of transitions from i to j should be equal to the expected number of transitions from j to i if the chain is reversible. On the other hand if it is possible to find a non-zero solution to $\pi_j p_{ji} = \pi_i p_{ij}$ then the vector $\vec{\pi}$ will be the unique stationary probability vector (Theorem 17 p. 182).

Whenever the graph of an irreducible Markov chain is a tree the chain will be reversible and we can apply $\pi_j p_{ji} = \pi_i p_{ij}$. Kolmogorovs criterion (Theroem 18 p. 185) give necessary and sufficient conditions for reversibility. The criterion is somewhat clumsy to apply.