Note on the Markovian Arrival Process for 04141 Stochastic Processes

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Contents

Chapter 1

The Markovian Arrival Process

The Markovian arrival Process (MAP) is an extremely versatile modelling tool in the theory of point processes. Although the abstract theory of point processes is fairly extensive and well developed see eg. [1] the MAP is the only fairly general model which can be treated analytically. Most expressions for point processes can be obtained in closed form for the MAP. Obviously due to the complexity in modelling quite general phenomena these formulae are generally quite complex expressions but their computer implementation is straightforward. Although the main virtue of the MAP is that many models can be analyzed explicitly it can be applied efficiently in simulation studies too.

The MAP was introduced by M.F. Neuts [6] as the versatile Markovian Process but later redefined as the MAP [3]. There is already a substantial litterature on the MAP and especially of its applications in queueing theory.

Before we introduce the MAP in its full generality we will examine one of the important special cases; the renewal process with interval distribution of phase type.

1.1 The PH renewal process

We consider a renewal process with PH distributed inter event times, $X_i \in \mathrm{PH}(\vec{\alpha}, \mathbf{T}), i > 1$. We have already treated this model in the note on phase type distributions. We will denote the stochastic process of the phases by $J(t)$. That is $J(t)$ takes the value of the phase or state in the Markov chain related to X_i. From the note on PH distributions we know that the mean of the inter event times is given by $E(X_i) = \mu_1 = \vec{\alpha}(-\mathbf{T})^{-1}\vec{e}$. There is a neat probabilistic interpretation of $(-\mathbf{T})^{-1}$ which we will discuss in the following section.

1.1.1 Probabilistic interpretation of $(-\mathbf{T})^{-1}$

We consider a PH-distributed interval X. Let us define τ_{ij} as the time spent in phase j conditioning on the event that the phase at time 0 is i. Further define the stochastic process

$$I_{ij}(t) = \begin{cases} 1 & \text{if } J(t) = j \\ 0 & \text{otherwise} \end{cases}$$

We then have $\tau_{ij} = \int_0^\infty I_{ij}(t)dt$ thus

$$E(\tau_{ij}) = E\left(\int_0^\infty I_{ij}(t)dt\right) = \int_0^\infty p_{ij}(t)dt$$

$$\int_0^K e^{\mathbf{T}t}dt = \int_0^K \sum_{i=0}^\infty \frac{(\mathbf{T}t)^i}{i!}dt = \sum_{i=0}^\infty \mathbf{T}^i \int_0^K \frac{t^i}{i!}dt = \sum_{i=0}^\infty \mathbf{T}^i \left[\frac{t^{i+1}}{(i+1)!}\right]_0^K = \tag{1.1}$$

$$\sum_{i=0}^\infty \mathbf{T}^i \frac{K^{i+1}}{(i+1)!} = \mathbf{T}^{-1}\left(e^{\mathbf{T}K} - \mathbf{I}\right) \rightarrow (-\mathbf{T})^{-1}$$

Thus the element (i, j) in $(-\mathbf{T})^{-1}$ is the expected time spent in phase j before absorption conditioned on the fact that the chain was started in phase i. It is clear from this probabilistic interpretation that $(-\mathbf{T})^{-1} \geq \mathbf{0}$. We can now get the mean time before absorption conditioning on start in i by taking row sums of $(-\mathbf{T})^{-1}$. Thus the i'th element of $(-\mathbf{T})^{-1}\vec{e}$ is the mean time spent in the transient states conditioning on start in i. To obtain the mean for a PH distribution with initial probability vector $\vec{\alpha}$ i.e. $\mu_1 = \vec{\alpha}(-\mathbf{T})^{-1}\vec{e}$ we will have to make a weighted sum of $(-\mathbf{T})^{-1}\vec{e}$ with $\vec{\alpha}$ as weighting factors. In the PH note this result was derived purely analytically by taking the derivative of Laplace transform. The vector $\vec{\pi}$ is the steady state probability vector of the Markov chain underlying the PH renewal process. In the note on PH distributions this vector is given by the expression $\vec{\pi} = \frac{1}{\mu_1}\vec{\alpha}(-\mathbf{T})^{-1}$. We are now able to interpret this result probabilistically. The i'th element of the j'th column of $(-\mathbf{T})^{-1}$ is the mean time spent in phase j with start in phase i. The product of $\vec{\alpha}$ with this column is thus the mean time spent in phase j by the phase time distribution with representation $(\vec{\alpha}, \mathbf{T})$. By inspection the process at a random time epoch the probability of finding the Markov chain in phase j is thus proportional to the mean time spent in phase j. The normalizing condition is obviously μ_1.

The example illustrates that substantial probabilistic interpretation can be deduced from formulae which has been derived by analytical considerations one of the appealing features of PH distributions and the Markovian arrival process Conversely a number of formulae can be derived more or less based on probabilistic insight.

Example 1.1 : The basic example of the PH renewal process is the Poisson process.

$$\mathbf{T} = \left[\ -\lambda\ \right] \qquad \vec{T}^0 = \left[\ \lambda\ \right] \qquad \vec{\alpha} = (1) \tag{1.2}$$

□

Example 1.2 : The renewal process with Erlang-2 distributed inter-event times.

$$\mathbf{T} = \begin{bmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{bmatrix} \qquad \vec{T^0} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \qquad \vec{\alpha} = (1,0) \tag{1.3}$$

we get

$$\vec{T^0}\vec{\alpha} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} (1,0) = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

The matrix of transitions associated with arrivals $\vec{T^0}\vec{\alpha}$ is

$$\vec{T^0}\vec{\alpha} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}$$

The generator matrix is given by

$$\mathbf{T} + \vec{T^0}\vec{\alpha} = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

$$(-\mathbf{T})^{-1} = \begin{bmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{bmatrix}^{-1} = \frac{1}{\lambda^2} \begin{bmatrix} \lambda & \lambda \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & \frac{1}{\lambda} \end{bmatrix}$$

And we find the mean $\frac{2}{\lambda}$. The vector $\vec{\pi} = \frac{\lambda}{2}(\frac{1}{\lambda}, \frac{1}{\lambda}) = (\frac{1}{2}, \frac{1}{2})$. Thus considering a renewal process with Erlang-2 distributed intervals the probability of finding the process in either state at an arbitrary time is $\frac{1}{2}$. The excess, the time until the next event, from a random time epoch is thus PH distributed with $(\vec{\pi}, \mathbf{T})$. The distribution is a mixture of an Erlang-2 distribution and an exponential, ie

$$f(t) = \frac{\lambda}{2}(1+\lambda t)e^{-\lambda t} \qquad F(t) = 1 - (1 + \frac{\lambda t}{2})e^{-\lambda t}$$

□

Example 1.3 : The renewal process with Hyperexponential distributed inter-event times.

$$\mathbf{T} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \qquad \vec{T^0} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad \vec{\alpha} = (p_1, p_2) \qquad p_1 + p_2 = 1 \tag{1.4}$$

We find the generator for the Markov changes of the phases

$$\mathbf{T} + \vec{T^0}\vec{\alpha} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 p_1 & \lambda_1 p_2 \\ \lambda_2 p_1 & \lambda_2 p_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 p_2 & \lambda_1 p_2 \\ \lambda_2 p_1 & -\lambda_2 p_1 \end{bmatrix}$$

Further we find

$$(-\mathbf{T})^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$$

and the mean is finally obtained as $\mu_1 = \frac{p_1}{\lambda_1} + \frac{p_2}{\lambda_2}$. With $\vec{\theta} = \frac{1}{\mu_1}\left(\frac{p_1}{\lambda_1}, \frac{p_2}{\lambda_2}\right)$ and once again we see that the excess from an arbitrary time epoch is again hyperexponential. □

1.2 Definition of the Markovian Arrival Process

By considering the construction of the PH renewal process we note that the generator matrix of the CTMC is split in two matrices $\mathbf{T}$ and $\vec{T}^0\vec{\alpha}$. The first matrix is related to phase transistions not associated with arrivals while transitions in the second are related to arrivals, $i-i$ transitions are obviously allowed in this setting. It is now tempting to generalize this construction such that we consider the more general possibilities in the splitting $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1$ where $\mathbf{D}$ is an irreducible generator of an m-state CTMC. Thus the PH renewal process is a MAP with $\mathbf{D}_0 = \mathbf{T}$ and $\mathbf{D}_1 = \vec{T}^0\vec{\alpha}$.

The matrix $\mathbf{D}_0$, associated with transitions without arrivals, is non-singular with negative diagonal elements and non-negative off diagonal elements. The matrix $\mathbf{D}_1$ associated with both transitions and arrivals is non-negative.

Example 1.4 : The Markov Modulated Poisson Process has been a popular tool for modelling phenomena which are inadequately described by a Poisson process. What characterizes the MMPP is that the matrix $\mathbf{D}_1$ is diagonal. This leads to substantial analytic simplifications. □

Example 1.5 : The most commonly applied and the simplest example of a MAP which is not a renewal process is the MAP with parameter matrices

$$\mathbf{D}_0 = \begin{bmatrix} -(\sigma_1+\gamma_1) & \sigma_1 \\ \sigma_2 & -(\sigma_2+\gamma_2) \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -\sigma_1 & \sigma_1 \\ \sigma_2 & -\sigma_2 \end{bmatrix}$$

This process which is the two dimensional version of the MMPP is called the switched Poisson process (SPP) and . Whenever the process is in state i we get arrivals according to a Poisson process with intensity γ_i. The sojourn time in each of the states is exponential with mean $\frac{1}{\sigma_i}$. □

Example 1.6 : The SPP with either $\gamma_1 = 0$ or $\gamma_2 = 0$ is called an Interrupted Poisson Process (IPP). This process is a renewal process (why?) It can be shown that the renewal process induced by the IPP is stochastically equivalent to a renewal process with hyperexponentially distributed intervals. The parameters of the IPP expressed in terms of the parameters of the hyperexponential distribution are

$$\gamma_1 = p_1\lambda_1 + p_2\lambda_2 \qquad \gamma_2 = 0 \tag{1.5}$$

$$\sigma_1 = \frac{(\lambda_1-\lambda_2)^2}{\frac{\lambda_1}{p_2}+\frac{\lambda_2}{p_1}} \tag{1.6}$$

$$\sigma_2 = \frac{1}{\frac{p_1}{\lambda_2}+\frac{p_2}{\lambda_1}} \tag{1.7}$$

□

With some experience we can design MAPs to model certain stochastic behaviour qualitatively.

Example 1.7 : As an example let us consider an alternating sequence of intervals with less respectively higher variability of the Poisson process. Intervals of the first kind can be modelled by an Erlang-2 distribution while intervals of the second kind can be modelled by a hyperexponential distribution. The expense we pay is the dimension of the MAP. The MAP outlined above has parameter matrices

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda p_1 & \lambda p_2 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \end{bmatrix}$$

□

As with PH distributions the class of MAPs is closed under a certain number of operations on point processes. A superposition of the two independent MAPs with parameters $(\mathbf{D}_0, \mathbf{D}_1)$ and $(\mathbf{C}_0, \mathbf{C}_1)$ is a MAP with parameter matrices $(\mathbf{D}_0 \bigoplus \mathbf{C}_0, \mathbf{D}_1 \bigoplus \mathbf{C}_1)$. A random thinning of a MAP $(\mathbf{D}_0, \mathbf{D}_1)$, that is a MAP where an event is kept with probability p, is itself a MAP with parameter matrices $(\mathbf{D}_0 + (1-p)\mathbf{D}_1, p\mathbf{D}_1)$.

Example 1.8 : The superposition of two identical SPPs of example 1.5 is a MAP $(\mathbf{C}_0, \mathbf{C}_1)$

$$\begin{aligned} \mathbf{C}_0 &= \begin{bmatrix} -2(\lambda_1+\sigma_1) & \sigma_1 & \sigma_1 & 0 \\ \sigma_2 & -(\lambda_1+\lambda_2+\sigma_1+\sigma_2) & 0 & \sigma_1 \\ \sigma_2 & 0 & -(\lambda_2+\lambda_1+\sigma_2+\sigma_1) & \sigma_1 \\ 0 & \sigma_2 & \sigma_2 & -2(\lambda_2+\sigma_2) \end{bmatrix} \\ \mathbf{C}_1 &= \begin{bmatrix} 2\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1+\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2+\lambda_1 & 0 \\ 0 & 0 & 0 & 2\lambda_2 \end{bmatrix} \end{aligned} \tag{1.8}$$

□

1.2.1 The counting process

The probabilities $\mathbf{P}(n,t)$

We derive the following system of equations, i.e. the forward Chapman-Kolmogorov equations

$$\mathbf{P}'(0,t) = \mathbf{P}(0,t)\mathbf{D}_0$$

$$\begin{aligned}
\mathbf{P}'(1,t) &= \mathbf{P}(0,t)\mathbf{D}_1 + \mathbf{P}(1,t)\mathbf{D}_0 \\
&\vdots \\
\mathbf{P}'(n+1,t) &= \mathbf{P}(n,t)\mathbf{D}_1 + \mathbf{P}(n+1,t)\mathbf{D}_0
\end{aligned} \tag{1.9}$$

It is well known that a system of linear differential equation can be effectively solved by use of the Laplace transform. For a system of difference equations as we are dealing with here, the z-transform is the corresponding tool. By multiplying the i'th equation by z^i we obtain

$$\begin{aligned}
\mathbf{P}'(0,t) &= \mathbf{P}(0,t)\mathbf{D}_0 \\
z\mathbf{P}'(1,t) &= z\mathbf{P}(0,t)\mathbf{D}_1 + z\mathbf{P}(1,t)\mathbf{D}_0 \\
&\vdots \\
z^{n+1}\mathbf{P}'(n+1,t) &= z^{n+1}\mathbf{P}(n,t)\mathbf{D}_1 + z^{n+1}\mathbf{P}(n+1,t)\mathbf{D}_0
\end{aligned} \tag{1.10}$$

summing these equations

$$\sum_{i=0}^{\infty} z^i\mathbf{P}'(0,t) = z\sum_{i=0}^{\infty} z^i\mathbf{P}(0,t)\mathbf{D}_1 + \sum_{i=0}^{\infty} z^i\mathbf{P}(0,t)\mathbf{D}_0 \tag{1.11}$$

Defining $\mathbf{P}(z,t) = \sum_{i=0}^{\infty} z^i\mathbf{P}(n,t)$ and $\mathbf{D}(z) = \mathbf{D}_0 + z\mathbf{D}_1$ and noting that $\sum_{i=0}^{\infty} z^i\mathbf{P}'(n,t) = \mathbf{P}'(z,t)$ we get

$$\mathbf{P}'(z,t) = \mathbf{P}(z,t)\mathbf{D}(z) \tag{1.12}$$

and finally we obtain

$$\mathbf{P}(z,t) = e^{\mathbf{D}(z)t} = e^{(\mathbf{D}_0 + z\mathbf{D}_1)t} \tag{1.13}$$

Recall that for the Poisson process with intensity parameter λ, $E\left(z^{N(t)}\right) = e^{-\lambda(1-z)t}$. Thus in the transform domain we see that the MAP is a very natural generalization of the Poisson process. It should be stressed however, that the distribution $\mathbf{P}(n,t)$ generally is very complicated. This is true even for the most simple generalisations of the Poisson process as the IPP or the Erlang 2 renewal process.

1.3 Moments of the Markovian Arrival Process

We can find the expected number of arrivals during an interval of length t by taking derivatives in (1.13).

$$\begin{aligned}
\frac{d}{dz}\mathbf{P}(z,t) = \frac{d}{dz}\sum_{i=0}^{\infty}\frac{(\mathbf{D}(z)t)^i}{i!} &= \frac{d}{dz}\sum_{i=0}^{\infty}\frac{((\mathbf{D}_0 + z\mathbf{D}_1)t)^i}{i!} = \\
\sum_{i=1}^{\infty}\frac{t^i}{i!}&\sum_{j=0}^{i-1}(\mathbf{D}_0 + z\mathbf{D}_1)^j\mathbf{D}_1(\mathbf{D}_0 + z\mathbf{D}_1)^{i-1-j} \\
\left.\frac{d}{dz}\mathbf{P}(z,t)\right|_{z=1} &= \sum_{i=1}^{\infty}\frac{t^i}{i!}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_1\mathbf{D}^{i-1-j}
\end{aligned} \tag{1.14}$$

The i, j'th element of this matrix is the expected number of $N(t)I_{ij}(t)$ i.e. the expected number of events in the interval $)0;t]$ if the phase is j at time t. Taking row sums by postmultiplying with $\vec{e}$ using $\mathbf{D}\vec{e} = \vec{0}$ we get

$$\left.\frac{d}{dz}\mathbf{P}(z,t)\right|_{z=1} \vec{e} = \vec{\mu}(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \mathbf{D}^{i-1}\mathbf{D}_1\vec{e} \tag{1.15}$$

The j'th element of $\vec{\mu}$ is the expected number of event in $)0;t]$ conditioning on phase j at time 0. The final result for the stationary MAP is obtained by

$$\vec{\theta}\vec{\mu}(t) = \vec{\theta}\mathbf{D}_1\vec{e}t = \lambda^{\star}t \tag{1.16}$$

since $\vec{\theta}\mathbf{D} = \vec{0}$. The value $\lambda^{\star}$ is called the fundamental rate of the MAP. Let us now continue with the expression for $\vec{\mu}$

$$\vec{\mu}(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathbf{D})^{i-1}\mathbf{D}_1\vec{e} \tag{1.17}$$

The expression has some flavour of the matrix exponential, the exponent of $\mathbf{D}$ is not i however. We cannot correct this fact in the straightforward manner due to the fact that $\mathbf{D}$ is singular. Since we know that $\vec{\theta}$ is unique up to a constant the eigenvalue at 0 has multiplicity 1. By using matrix theoretic arguments it can be shown that the matrix $\vec{e}\vec{\theta} - \mathbf{D}$ is nonsingular. We will reuse this property substantially in the following thus introducing the symbol $\mathbf{\Theta} = \vec{e}\vec{\theta}$. To explain the trick loosely; the matrix $\mathbf{D}$ has a single eigenvalue of 0. By constructing the matrix $\mathbf{\Theta} - \mathbf{D}$ we move that eigenvalue from zero to one without altering the rest of the eigenstructure, i.e. all $\mathbf{D}$ and $\mathbf{\Theta} - \mathbf{D}$ agree on all other eigenvalues and on all eigenvectors.

$$\vec{\theta}\left(\mathbf{\Theta} - \mathbf{D}\right) = \vec{\theta}\vec{e}\vec{\theta} - \vec{\theta}\mathbf{D} = \vec{\theta}$$

It is seen that $\vec{\theta}$ is still an eigenvector now corresponding to an eigenvalue with value 1. Since all other left eigenvectors of $\mathbf{D}$ than $\vec{\theta}$ are orthogonal on $\vec{e}$ they will all be eigenvalues of the new matrix too. Consider an arbitrary such eigenvector $\vec{w}$ with eigenvalue η

$$\vec{w}\left(\vec{e}\vec{\theta} - \mathbf{D}\right) = \vec{w}\vec{e}\vec{\theta} - \vec{w}\mathbf{D} = -\eta\vec{w}$$

We will rewrite the expression for $\vec{\mu}(t)$ using this new insight

$$\begin{aligned}\vec{\mu}(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathbf{\Theta} - \mathbf{D})^{-1}(\mathbf{\Theta} - \mathbf{D})\mathbf{D}^{i-1}\mathbf{D}_1\vec{e} = \\ (\mathbf{\Theta} - \mathbf{D})^{-1}\sum_{i=1}^{\infty} \frac{t^i}{i!}(\vec{e}\vec{\theta} - \mathbf{D})\mathbf{D}^{i-1}\mathbf{D}_1\vec{e}\end{aligned} \tag{1.18}$$

The products $\vec{e}\vec{\theta}\mathbf{D}^{i-1}$ will obviously vanish whenever $i > 1$ so we can rewrite into

$$\begin{aligned}(\vec{e}\vec{\theta} - \mathbf{D})^{-1}\left(t\vec{e}\vec{\theta}\mathbf{D}_1\vec{e} - \sum_{i=1}^{\infty}\frac{t^i}{i!}\mathbf{D}^i\mathbf{D}_1\vec{e}\right) = \\ (\vec{e}\vec{\theta} - \mathbf{D})^{-1}\left(t\vec{e}\vec{\theta}\mathbf{D}_1\vec{e} - \sum_{i=1}^{\infty}\frac{t^i}{i!}\mathbf{D}^i\mathbf{D}_1\vec{e}\right) = \\ (\vec{e}\vec{\theta} - \mathbf{D})^{-1}t\vec{e}\vec{\theta}\mathbf{D}_1\vec{e} + (\vec{e}\vec{\theta} - \mathbf{D})^{-1}(\mathbf{I} - e^{\mathbf{D}t})\vec{e}\end{aligned} \tag{1.19}$$

Finally we note that since $\vec{e}$ is a right eigenvector of $(\mathbf{\Theta}-\mathbf{D})$ it is also a right eigenvector of $(\mathbf{\Theta}-\mathbf{D})^{-1}$

$$\vec{\mu}(t) = t\mathbf{\Theta}\mathbf{D}_1\vec{e} + (\mathbf{\Theta}-\mathbf{D})^{-1}(\mathbf{I}-e^{\mathbf{D}t})\vec{e} = t\mathbf{\Theta}\mathbf{D}_1\vec{e} + (\mathbf{I}-e^{\mathbf{D}t})(\mathbf{\Theta}-\mathbf{D})^{-1}\vec{e} \tag{1.20}$$

Last equality is due to the commutativity of the matrices involved. Once again we can verify that for the stationary MAP $\vec{\theta}\vec{\mu} = \vec{\theta}\mathbf{D}_1\vec{e} = \lambda^\star$. This result is of course not surprising since it is a general property of a stationary point process. The versatility of the MAP allows us to calculate explicitly $E(N(t))$ for any other initial probability distribution of the phases.

1.3.1 Second order of count descriptors

In order to derive expressions for $E(N^2(t))$ and thus $Var(N(t))$ we take second derivatives in (1.13) and basically apply the same matrix operations and reuse the analytical tricks that we used for first order properties. The calculus is somewhat tedious so we will state formulas from now on without proof. Hopefully, however, it should be clear from the preceeding section that the calculation of descriptors for the MAP is straightforward in principle although in practice a certain skill with respect to matrix algebra is needed.

Correlation of number of counts

The stochastic variable $N_{[t,t_1[}$ denotes the number of counts (arrivals or encounters) in the semiopen interval from t to t_1. The correlation in the counting process (again from [4], p.778 (36)) is

$$E[(N_{[0,t[} - E(N_{[0,t[}))(N_{[t+t_1,t+t_1+t_2[} - E(N_{[t+t_1,t+t_1+t_2[}))] = \\ \vec{\theta}\mathbf{D}_1\left(\mathbf{I}-\mathbf{e}^{\mathbf{Dt}}\right)e^{\mathbf{Dt_1}}\left(\mathbf{I}-\mathbf{e}^{\mathbf{Dt_2}}\right)(\mathbf{\Theta}-\mathbf{D})^{-2}\mathbf{D_1}\vec{\mathbf{e}} \tag{1.21}$$

Index of dispersion of counts

This expression is usually only calculated for the time stationary process. It is possible though make an expression for any initial probability distribution. For the time stationary case we obtain

$$(1-2\lambda^\star) + 2\vec{\theta}\mathbf{D}_1(\mathbf{\Theta}-\mathbf{D})^{-1}\mathbf{D}_1\vec{e}(\lambda^\star)^{-1} - 2\vec{\theta}\mathbf{D}_1(\mathbf{I}-e^{\mathbf{D}})(\mathbf{\Theta}-\mathbf{D})^{-2}\mathbf{D}_1\vec{e}(\lambda^\star t)^{-1}$$

The rate process

The MAP is essentially a double stochastic Poisson proces. The rate has a finite sample space of the row sum of the matrix of arrival intensities, i.e. attains the values of the vector $\mathbf{D}_1\vec{e} = \vec{\lambda}$. The mean of the rate process is nothing but the fundamental rate $\lambda^\star$ while the correlation function is given by ([8] p.14 formula (2.7))

$$\gamma(h) = \frac{\vec{\theta}\mathbf{D}_1\left(e^{\mathbf{D}h}\vec{\lambda}\right) - (\lambda^\star)^2}{\vec{\theta}(\vec{\lambda}\circ\vec{\lambda}) - (\lambda^\star)^2}$$

Where $\circ$ is the entrywise or Schur product ie. $\vec{\lambda}\circ\vec{\lambda} = (\lambda_1^2, \lambda_2^2, \ldots, \lambda_m^2)$.

Square Wave Spectral Density

See [7, 5] for a definition and a thoruogh discussion. This measure has been proposed by Neuts but has not gained any popularity in the litterature yet.

$$S(f) = \vec{\theta}\mathbf{D}_1\left[4\pi^2 f^2\mathbf{I} + \mathbf{D}_0 - \mathbf{D}_1)^2\right]^{-1}\vec{e} \tag{1.22}$$

1.3.2 Higher order information on counts

We can get higher order information by taking successive derivatives in (1.13). This is straightforward in principle but rather cumbersome in practice.

1.3.3 The PH renewal process

There are no substantial simplifications in the MAP formulas for the equilibrium or time stationary version of the PH renewal process. We will state some results for the ordinary renewal process, that is the version where X_1 has the same PH distribution $(\vec{\alpha}, \mathbf{T})$ as $X_i, i > 1$.

The renewal density

The renewal density is $m'(t)$ and $m'(t)\Delta t + o(\Delta t)$ is the probability of an event in the interval $)t; t + \Delta t$. We state without proof for the ordinary PH renewal process

$$m'(t) = \vec{\alpha}e^{\mathbf{D}t}\vec{T}^0 \tag{1.23}$$

For $\vec{\alpha} = \vec{\theta}$ this expressions is obviously $\vec{\theta}\vec{T}^0$.

Example 1.9 : For the Erlang-2 renewal process we find

$$m_o(t) = \frac{\lambda}{2}t + \frac{1}{4}\left(e^{-2\lambda t} - 1\right) \tag{1.24}$$

$$Var(N_e(t)) = \frac{\lambda t}{4} + \frac{1}{8}\left(1 - e^{-2\lambda t}\right) \tag{1.25}$$

$$Var(N_o(t)) = \frac{\lambda t}{4} + \frac{1}{16} - \frac{\lambda t}{2}e^{-2\lambda t} - \frac{1}{16}e^{-4\lambda t} \tag{1.26}$$

□

Example 1.10 : For the renewal process with hyperexponential interval distribution we find

$$m_o(t) = \frac{t}{\mu_1} + \frac{\sigma^2 - \mu_1^2}{2\mu_1^2} - \frac{p_1 p_2 \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2}{\mu_1^2} e^{-(p_1\lambda_2 + p_2\lambda_1)t} \tag{1.27}$$

Here the variance of the distribution $\mu_2 - \mu_1^2$ is denoted by σ^2. □

1.4 Interval stationary Process

Usually the main emphasis is on the time stationary version of the MAP, ie the process where the phase at time 0 is chosen according to $\vec{\theta}$. When the MAP is a PH renewal process this corresponds to examining the equilibrium renewal process. For the PH renewal process we have considered the ordinary version in which case all intervals are distributed with the same distribution. What we have to do is to chose the first interval as a typical interval. We find such an interval by considering the embedded Markov chain immediately after arrivals or epochs of events. The embedded Markov chain is a Markov chain in discrete time with transition probability matrix

$$\mathbf{P} = (-\mathbf{D}_0)^{-1}\mathbf{D}_1 \tag{1.28}$$

Some of the states in the discrete time Markov chain might be ephemeral. An ephemeral state is a state which can not be entered from any other state.

The event stationary probability vector $\vec{\phi}$

$$\vec{\phi} = \frac{\vec{\theta}\mathbf{D}_1}{\vec{\theta}\mathbf{D}_1\vec{e}} = \frac{\vec{\theta}\mathbf{D}_1}{\lambda^\star} \tag{1.29}$$

Since some of the states in the EMC can be ephemeral some of the entries of $\vec{\phi}$ can be zero.

Distribution of time between arrivals

The interevent distributions in the MAP are of phase type. The time from an arbitrary time to the first event has the PH representation $(\vec{\theta}, \mathbf{D}_0)$, the distribution of an arbitrary interval has the representation $(\vec{\phi}, \mathbf{D}_0)$.

Example 1.11 : The MAP with the parameter matrices

$$\mathbf{D}_0 = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\mu \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{bmatrix}$$

is an alternating renewal process. The alternating intervals are exponentially distributed respectively generalized Erlang-2 distributed. We find

$$(-\mathbf{D}_0)^{-1} = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & \frac{1}{\mu} \\ 0 & 0 & \frac{1}{\mu} \end{bmatrix}$$

and

$$(-\mathbf{D}_0)^{-1}\mathbf{D}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

State 3 is ephemeral. Further

$$\vec{\theta} = \left(\frac{2}{\lambda} + \frac{1}{\mu}\right)^{-1} \left(\frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\mu}\right) \qquad \vec{\phi} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

The Laplace transform of the distribution of an arbitrary inter arrival time is

$$\tilde{H}(s) = \frac{1}{2}\frac{\lambda}{s+\lambda} + \frac{1}{2}\frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu} = \frac{\frac{\lambda}{2}}{s+\lambda}\left(1 + \frac{\mu}{s+\mu}\right) = \frac{\frac{\lambda}{2}(s+2\mu)}{(s+\lambda)(s+\mu)}$$

If we $\lambda = 2\mu$ we find this distribution to be $\frac{\mu}{s+\mu}$ the Laplace transform of an exponential distribution. Thus the time between two arbitrary arrivals is exponential. Nevertheless, the process is not a Poisson process! □

Correlation of intervals

The covariance between intervals seperated by $k-1$ intervals (event stationary) are given by ([2] p.153 formula (19) and further simplifications)

$$(\lambda^\star)^{-1}\vec{\theta}\left(\left((-\mathbf{D}_0)^{-1}\mathbf{D}_1\right)^k - \mathbf{\Phi}\right)(-\mathbf{D}_0)^{-1}\vec{e} \tag{1.30}$$

with $\mathbf{\Phi} = \vec{e}\vec{\phi}$.

Index of dispersion of intervals

The index of dispersion of intervals is k times the ratio between the variance and the ssquared mean of the total time to the k'th event.

$$2\frac{1}{\vec{\theta}\mathbf{D}_1\vec{e}}\vec{\theta}(\mathbf{I}+\mathbf{D}_0^{-1}\mathbf{D}_1+\mathbf{\Phi})^{-1}$$

$$\left\{n\left[\mathbf{I}+\frac{n-1}{2}\mathbf{\Phi}\right]-(\mathbf{I}+\mathbf{D}_0^{-1}\mathbf{D}_1+\mathbf{\Phi})^{-1}(-\mathbf{D}_0^{-1}\mathbf{D}_1)(\mathbf{I}-(-\mathbf{D}_0^{-1}\mathbf{D}_1)^n)\right\}(-\mathbf{D}_0)^{-1}\vec{e} \tag{1.31}$$

1.5 The MAP as a Markov renewal process

The MAP is a Markov renewal process. To illustrate this we will make a continuation of example 1.11

Example 1.12 : The MAP of example 1.11 can be formulated as a Markov renewal process with the kernel

$$\mathbf{A}(t)=\left[\begin{array}{cc}0 & 1-e^{-\lambda t}\\ 1-\frac{\lambda}{\lambda-\mu}e^{-\mu t}-\frac{\mu}{\mu-\lambda}e^{-\lambda t} & 0\end{array}\right]=\mathbf{H}(t)\mathbf{P} \tag{1.32}$$

with

$$\mathbf{H}(t)=\left[\begin{array}{cc}1-e^{-\lambda t} & 0\\ 0 & 1-\frac{\lambda}{\lambda-\mu}e^{-\mu t}-\frac{\mu}{\mu-\lambda}e^{-\lambda t}\end{array}\right]\qquad \mathbf{P}=\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right] \tag{1.33}$$

□

The example indicates that the dimension of the Markov renewal is the number of non ephemeral states of the EMC after arrivals. This is actually true in general. We could formulate a Markov renewal process with the ephemeral states too, but this will not be relevant in general. One could question why the MAP formalism is introduced since the Markov renewal formulation is already well-established and is slightly more general than the MAP. The advantage of the MAP formalism is that it preserves the Markov property of the process not only at arrivals but more generally in time. The formulae stated in this note in MAP formalism are generally less transparent and less explicit if expressed in terms of the Markov renewal process.

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