$$
\begin{aligned}
\mathbb{P}(R_x = n) &= \boldsymbol{\beta} \boldsymbol{S}^{x+n-1}\boldsymbol{s} + \boldsymbol{\beta} \sum_{m=1}^{x} \left[ (\boldsymbol{S}+\boldsymbol{s\alpha})^m - (\boldsymbol{S}+\boldsymbol{s\alpha})^{m-1}\boldsymbol{S} \right] \boldsymbol{S}^{x-m+n-1}\boldsymbol{s} \\
&= \boldsymbol{\beta} \boldsymbol{S}^{x+n-1}\boldsymbol{s} + \boldsymbol{\beta} \sum_{m=1}^{x} \left[ (\boldsymbol{S}+\boldsymbol{s\alpha})^m \boldsymbol{S}^{-m} - (\boldsymbol{S}+\boldsymbol{s\alpha})^{m-1}\boldsymbol{S}^{-(m-1)} \right] \boldsymbol{S}^{x+n-1}\boldsymbol{s} \\
&= \boldsymbol{\beta} \boldsymbol{S}^{x+n-1}\boldsymbol{s} + \boldsymbol{\beta} \left[ (\boldsymbol{S}+\boldsymbol{s\alpha})^x \boldsymbol{S}^{n-1} - \boldsymbol{S}^{x+n-1} \right] \boldsymbol{s} \\
&= \boldsymbol{\beta} (\boldsymbol{S}+\boldsymbol{s\alpha})^x \boldsymbol{S}^{n-1}\boldsymbol{s}.
\end{aligned}
$$

□

An alternative proof of Theorem 5.7.21 is outlined in Problem 5.7.29, p. 359.

## Problems

**5.7.22.** The company A/S Satellite has some electronic control equipment that is stationed on Earth. A critical component of this system has a lifetime distribution given by the Laplace transform $L(\theta)$:

$$L(\theta) = \frac{2}{3}\frac{1}{\theta+1} + \frac{2}{3}\frac{1}{(\theta+1)^2+1}.$$

The time unit is one month. A component of this type is replaced immediately with a new one on failure.

(a) Calculate the mean and variance in the lifetime distribution of the components.

(b) Determine the probability that 30 or more components have to be replaced during a time span of two years.

(c) Give an exact expression for the expected number of replacements during a time interval of length $t$, under the assumption that a new component was installed at $t = 0$.

At some point, the information on the time in service of a component currently in service is lost.

(d) Give an expression for the distribution of the remaining lifetime of the component.

(e) Give an expression for the expected number of components that will be replaced during a time interval of length $t$.

**5.7.23.** Cars arrive at a highway rest area according to a Poisson process with intensity $\lambda$. The number of persons in a car can be described as being discrete phase-type distributed by a distribution with representation $(\alpha, \mathbf{T})$, whose parameters are given by

$$\alpha = \left(\frac{2}{3}, \frac{1}{3}\right) \qquad \mathbf{T} = \begin{bmatrix} p & q \\ 0 & p \end{bmatrix} \qquad q = 1-p.$$