It is worth emphasis that the p.g.f.s (3.115) and (3.116) have the same mean $p\phi'(1)$ but generally not the same variance, the first being

$$p\left[\phi''(1)+\phi'(1)-(\phi'(1))^2\right]$$

as compared with

$$p^2\phi''(1)+p\phi'(1)-p^2(\phi'(1))^2.$$

Example A second example leading to (3.116), as opposed to (3.115), concerns the different forms of mortality that affect a population. We appraise the strength (*stability*) of a population as the probability of indefinite survivorship = 1− probability of eventual extinction.

In the absence of mortality, the offspring number X of a single individual has the p.g.f. $\phi(s)$. Assume, consistent with the postulates of a branching process, that all offspring in the population behave independently governed by the same probability laws. Assume also an adult population of size $X = k$. We consider three types of mortality:

(a) *Mortality of Individuals* Let p be the probability of an offspring surviving to reproduce, independently of what happens to others. Thus, the contribution of each litter (family) to the adult population of the next generation has a binomial distribution with parameters (N, p), where N is the progeny size of the parent with p.g.f. $\phi(s)$. The p.g.f. of the adult numbers contributed by a single parent is, therefore, $\phi(q+ps), q = 1-p$, and for the population as a whole is

$$\psi_1(s) = [\phi(q+ps)]^k. \tag{3.117}$$

This type of mortality might reflect predation on adults.

(b) *Mortality of Litters* Independently of what happens to other litters, each litter survives with probability p and is wiped out with probability $q = 1-p$. That is, given an actual litter size ξ, the effective litter size is ξ with probability p, and 0 with probability q. The p.g.f. of adults in the following generation is accordingly

$$\psi_2(s) = [q+p\phi(s)]^k. \tag{3.118}$$

This type of mortality might reflect predation on juveniles or on nests and eggs in the case of birds.

(c) *Mortality of Generations* An entire generation survives with probability p and is wiped out with probability q. This type of mortality might represent environmental catastrophes (e.g., forest fire, flood). The p.g.f. of population size in the next generation in this case is

$$\psi_3(s) = q+p[\phi(s)]^k. \tag{3.119}$$

All the p.g.f.s (3.117) through (3.119) have the same mean but usually different variances.

It is interesting to assess the relative stability of these three models. That is, we need to compare the smallest positive roots of $\psi_i(s) = s, i = 1,2,3$, which we will denote by $s_i^*, i = 1,2,3$, respectively.

We will show by convexity analysis that

$$\psi_1(s) \le \psi_2(s) \le \psi_3(s).$$

A function $f(x)$ is convex in x if for every x_1 and x_2 and $0<\lambda<1$, then $f[\lambda x_1 + (1-\lambda)x_2] \le \lambda f(x_1)+(1-\lambda)f(x_2)$. In particular, the function $\phi(s) = \sum_{k=0}^{\infty} p_k s^k$ for $0<s<1$ is convex in s, since for each positive integer k, $[(\lambda s_1)+(1-\lambda)s_2]^k \le \lambda s_1^k + (1-\lambda)s_2^k$ for $0<\lambda, s_1, s_2<1$. Now, $\psi_1(s) = [\phi(q+ps)]^k < [q\phi(1)+p\phi(s)]^k = [q+p\phi(s)]^k = \psi_2(s)$, and then $s_1^* < s_2^*$. Thus, the first model is more stable than the second model.

Observe further that due to the convexity of $f(x) = x^k, x>0$, $\psi_2(s) = [p\phi(s)+q]^k < p[\phi(s)]^k + q\times 1^k = \psi_3(s)$, and thus $s_2^* < s_3^*$, implying that the second model is more stable than the third model. In conjunction we get the ordering $s_1^* < s_2^* < s_3^*$.

Exercises

3.9.1 Suppose that the offspring distribution is Poisson with mean $\lambda = 1.1$. Compute the extinction probabilities $u_n = \Pr\{X_n = 0|X_0 = 1\}$ for $n = 0,1,\dots,5$. What is u_∞, the probability of ultimate extinction?

3.9.2 Determine the probability generating function for the offspring distribution in which an individual either dies, with probability p_0, or is replaced by two progeny, with probability p_2, where $p_0+p_2 = 1$.

3.9.3 Determine the probability generating function corresponding to the offspring distribution in which each individual produces 0 or N direct descendants, with probabilities p and q, respectively.

3.9.4 Let $\phi(s)$ be the generating function of an offspring random variable ξ. Let Z be a random variable whose distribution is that of ξ, but conditional on $\xi > 0$. That is,

$$\Pr\{Z = k\} = \Pr\{\xi = k|\xi > 0\} \quad \text{for } k = 1,2,\dots.$$

Express the generating function for Z in terms of ϕ.

Problems

3.9.1 One-fourth of the married couples in a far-off society have no children at all. The other three-fourths of couples have exactly three children, with each child equally likely to be a boy or a girl. What is the probability that the male line of descent of a particular husband will eventually die out?

3.9.2 One-fourth of the married couples in a far-off society have exactly three children. The other three-fourths of couples continue to have children until the first boy and then cease childbearing. Assume that each child is equally likely to be a boy or girl. What is the probability that the male line of descent of a particular husband will eventually die out?

3.9.3 Consider a large region consisting of many subareas. Each subarea contains a branching process that is characterized by a Poisson distribution with parameter λ. Assume, furthermore, that the value of λ varies with the subarea, and its distribution over the whole region is that of a gamma distribution. Formally, suppose that the offspring distribution is given by

$$\pi(k|\lambda) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, \ldots,$$

where λ itself is a random variable having the density function

$$f(\lambda) = \frac{\theta^\alpha \lambda^{\alpha-1} e^{-\theta\lambda}}{\Gamma(\alpha)} \quad \text{for } \lambda > 0,$$

where θ and α are positive constants. Determine the marginal offspring distribution $p_k = \int \pi(k|\lambda) f(\lambda) d\lambda$.

Hint: Refer to the last example of Chapter 2, Section 2.4.

3.9.4 Let $\phi(s) = 1 - p(1-s)^\beta$, where p and β are constants with $0 < p, \beta < 1$. Prove that $\phi(s)$ is a probability generating function and that its iterates are

$$\phi_n(s) = 1 - p^{1+\beta+\cdots+\beta^{n-1}}(1-s)^{\beta^n} \quad \text{for } n = 1, 2, \ldots.$$

3.9.5 At time 0, a blood culture starts with one red cell. At the end of 1 min, the red cell dies and is replaced by one of the following combinations with the probabilities as indicated:

Two red cells	$\frac{1}{4}$
One red, One white	$\frac{2}{3}$
Two white	$\frac{1}{12}$

Each red cell lives for 1 min and gives birth to offspring in the same way as the parent cell. Each white cell lives for 1 min and dies without reproducing. Assume that individual cells behave independently.

(a) At time $n + \frac{1}{2}$ min after the culture begins, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture eventually dies out entirely?

3.9.6 Let $\phi(s) = as^2 + bs + c$, where a, b, c are positive and $\phi(1) = 1$. Assume that the probability of extinction is u_∞, where $0 < u_\infty < 1$. Prove that $u_\infty = c/a$.

3.9.7 Families in a certain society choose the number of children that they will have according to the following rule: If the first child is a girl, they have exactly one more child. If the first child is a boy, they continue to have children until the first girl and then cease childbearing. Let ξ be the number of male children in a particular family. What is the generating function of ξ? Determine the mean of ξ directly and by differentiating the generating function.

3.9.8 Consider a branching process whose offspring follow the geometric distribution $p_k = (1-c)c^k$ for $k = 0, 1, \ldots$, where $0 < c < 1$. Determine the probability of eventual extinction.

3.9.9 One-fourth of the married couples in a distant society have no children at all. The other three-fourths of couples continue to have children until the first girl and then cease childbearing. Assume that each child is equally likely to be a boy or girl.

(a) For $k = 0, 1, 2, \ldots$, what is the probability that a particular husband will have k male offspring?

(b) What is the probability that the husband's male line of descent will cease to exist by the fifth generation?

3.9.10 Suppose that in a branching process the number of offspring of an initial particle has a distribution whose generating function is $f(s)$. Each member of the first generation has a number of offspring whose distribution has generating function $g(s)$. The next generation has generating function f, the next has g, and the distributions continue to alternate in this way from generation to generation.

(a) Determine the extinction probability of the process in terms of $f(s)$ and $g(s)$.

(b) Determine the mean population size at generation n.

(c) Would any of these quantities change if the process started with the $g(s)$ process and then continued to alternate?