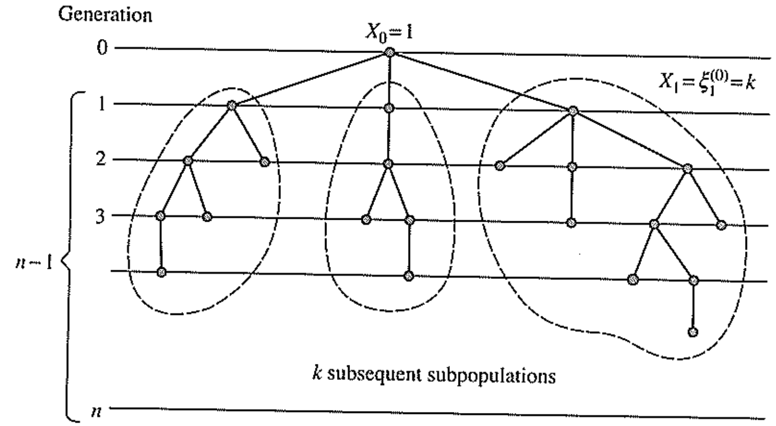

Figure 3.6 The diagram illustrates that if the original population is to die out by generation n, then the subpopulations generated by distinct initial offspring must all die out in $n-1$ generations.

Now, the k subpopulations generated by the distinct offspring of the original parent are independent, and they have the same statistical properties as the original population. Therefore, the probability that any particular one of them dies out in $n-1$ generations is u_{n-1} by definition, and the probability that all k subpopulations die out in $n-1$ generations is the kth power $(u_{n-1})^k$ because they are independent. Upon weighting this factor by the probability of k offspring and summing according to the law of total probability, we obtain

$$u_n = \sum_{k=0}^{\infty} p_k (u_{n-1})^k, \quad n = 1, 2, \ldots. \tag{3.101}$$

Of course $u_0 = 0$, and $u_1 = p_0$, the probability that the original parent had no offspring.

Example Suppose a parent has no offspring with probability $\frac{1}{4}$ and two offspring with probability $\frac{3}{4}$. Then, the recursion (3.101) specializes to

$$u_n = \frac{1}{4} + \frac{3}{4}(u_{n-1})^2 = \frac{1 + 3(u_{n-1})^2}{4}.$$

Beginning with $u_0 = 0$, we successively compute

$$\begin{aligned} u_1 &= 0.2500, & u_6 &= 0.3313,\\ u_2 &= 0.2969, & u_7 &= 0.3323,\\ u_3 &= 0.3161, & u_8 &= 0.3328,\\ u_4 &= 0.3249, & u_9 &= 0.3331,\\ u_5 &= 0.3292, & u_{10} &= 0.3332. \end{aligned}$$

We see that the chances are very nearly $\frac{1}{3}$ that such a population will die out by the tenth generation.

Exercises

3.8.1 A population begins with a single individual. In each generation, each individual in the population dies with probability $\frac{1}{2}$ or doubles with probability $\frac{1}{2}$. Let X_n denote the number of individuals in the population in the nth generation. Find the mean and variance of X_n.

3.8.2 The number of offspring of an individual in a population is $0, 1$, or 2 with respective probabilities $a > 0$, $b > 0$, and $c > 0$, where $a + b + c = 1$. Express the mean and variance of the offspring distribution in terms of b and c.

3.8.3 Suppose a parent has no offspring with probability $\frac{1}{2}$ and has two offspring with probability $\frac{1}{2}$. If a population of such individuals begins with a single parent and evolves as a branching process, determine u_n, the probability that the population is extinct by the nth generation, for $n = 1, 2, 3, 4, 5$.

3.8.4 At each stage of an electron multiplier, each electron, upon striking the plate, generates a Poisson distributed number of electrons for the next stage. Suppose the mean of the Poisson distribution is λ. Determine the mean and variance for the number of electrons in the nth stage.

Problems

3.8.1 Each adult individual in a population produces a fixed number M of offspring and then dies. A fixed number L of these remain at the location of the parent. These local offspring will either all grow to adulthood, which occurs with a fixed probability β, or all will die, which has probability $1-\beta$. Local mortality is catastrophic in that it affects the entire local population. The remaining $N = M - L$ offspring disperse. Their successful growth to adulthood will occur statistically independently of one another, but at a lower probability $\alpha = p\beta$, where p may be thought of as the probability of successfully surviving the dispersal process. Define the random variable ξ to be the number of offspring of a single parent that survive to reach adulthood in the next generation. According to our assumptions, we may write ξ as

$$\xi = \nu_1 + \nu_2 + \cdots + \nu_N + (M - N)\Theta,$$

where $\Theta, \nu_1, \nu_2, \ldots, \nu_N$ are independent with $\Pr\{\nu_k = 1\} = \alpha$, $\Pr\{\nu_k = 0\} = 1 - \alpha$, and with $\Pr\{\Theta = 1\} = \beta$ and $\Pr\{\Theta = 0\} = 1 - \beta$. Show that the mean number of offspring reaching adulthood is $E[\xi] = \alpha N + \beta(M - N)$, and since $\alpha < \beta$, the mean number of surviving offspring is maximized by dispersing none ($N = 0$). Show that the probability of having no offspring surviving to adulthood is

$$\Pr\{\xi = 0\} = (1-\alpha)^N (1-\beta)$$

and that this probability is made smallest by making N large.

3.8.2 Let $Z = \sum_{n=0}^{\infty} X_n$ be the total family size in a branching process whose offspring distribution has a mean $\mu = E[\xi] < 1$. Assuming that $X_0 = 1$, show that $E[Z] = 1/(1-\mu)$.

3.8.3 Families in a certain society choose the number of children that they will have according to the following rule: If the first child is a girl, they have exactly one more child. If the first child is a boy, they continue to have children until the first girl, and then cease childbearing.

(a) For $k = 0, 1, 2, \ldots$, what is the probability that a particular family will have k children in total?

(b) For $k = 0, 1, 2, \ldots$, what is the probability that a particular family will have exactly k male children among their offspring?

3.8.4 Let $\{X_n\}$ be a branching process with mean family size μ. Show that $Z_n = X_n/\mu^n$ is a nonnegative martingale. Interpret the maximal inequality as applied to $\{Z_n\}$.

3.9 Branching Processes and Generating Functions*

Consider a nonnegative integer-valued random variable ξ whose probability distribution is given by

$$\Pr\{\xi = k\} = p_k \quad \text{for } k = 0, 1, \ldots. \tag{3.102}$$

The *generating function* $\phi(s)$ associated with the random variable ξ (or equivalently, with the distribution $\{p_k\}$) is defined by

$$\phi(s) = E\left[s^{\xi}\right] = \sum_{k=0}^{\infty} p_k s^k \quad \text{for } 0 \le s \le 1. \tag{3.103}$$

Much of the importance of generating functions derives from the following three results.

First, the relation between probability mass functions (3.102) and generating functions (3.103) is one-to-one. Thus, knowing the generating function is equivalent, in some sense, to knowing the distribution. The relation that expresses the probability mass function $\{p_k\}$ in terms of the generating function $\phi(s)$ is

$$p_k = \frac{1}{k!} \left.\frac{\mathrm{d}^k\phi(s)}{\mathrm{d}s^k}\right|_{s=0}. \tag{3.104}$$

For example,

$$\phi(s) = p_0 + p_1 s + p_2 s^2 + \cdots,$$

whence

$$p_0 = \phi(0),$$

and

$$\frac{\mathrm{d}\phi(s)}{\mathrm{d}s} = p_1 + 2p_2 s + 3p_3 s^2 + \cdots,$$

whence

$$p_1 = \left.\frac{\mathrm{d}\phi(s)}{\mathrm{d}s}\right|_{s=0}.$$

Second, if $\xi_1, \ldots, \xi_n$ are independent random variables having generating functions $\phi_1(s), \ldots, \phi_n(s)$, respectively, then the generating function of their sum $X = \xi_1 + \cdots + \xi_n$ is simply the product

$$\phi_X(s) = \phi_1(s)\phi_2(s)\cdots\phi_n(s). \tag{3.105}$$

This simple result makes generating functions extremely helpful in dealing with problems involving sums of independent random variables. It is to be expected, then, that generating functions might provide a major tool in the analysis of branching processes.

Third, the moments of a nonnegative integer-valued random variable may be found by differentiating the generating function. For example, the first derivative is

$$\frac{\mathrm{d}\phi(s)}{\mathrm{d}s} = p_1 + 2p_2 s + 3p_3 s^2 + \cdots,$$

whence

$$\left.\frac{\mathrm{d}\phi(s)}{\mathrm{d}s}\right|_{s=1} = p_1 + 2p_2 + 3p_3 + \cdots = E[\xi], \tag{3.106}$$

and the second derivative is

$$\frac{\mathrm{d}^2\phi(s)}{\mathrm{d}s^2} = 2p_2 + 3(2)p_3 s + 4(3)p_4 s^2 + \cdots,$$

whence

$$\begin{aligned}\left.\frac{\mathrm{d}^2\phi(s)}{\mathrm{d}s^2}\right|_{s=1} &= 2p_2 + 3(2)p_3 + 4(3)p_4 + \cdots \\ &= \sum_{k=2}^{\infty} k(k-1)p_k = E[\xi(\xi-1)] \\ &= E\left[\xi^2\right] - E[\xi].\end{aligned}$$

* This topic is not prerequisite to what follows.