Equation (3.93) may be used in conjunction with (3.85) to verify the first step analysis equation for U_{ik}. We multiply (3.85) by R_{jk} and sum, obtaining thereby

$$\sum_{j=0}^{r-1} W_{ij}R_{jk} = \sum_{j=0}^{r-1} \delta_{ij}R_{jk} + \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} P_{il}W_{lj}R_{jk},$$

which with (3.93) gives

$$\begin{aligned} U_{ik} &= R_{ik} + \sum_{l=0}^{r-1} P_{il}U_{lk} \\ &= P_{ik} + \sum_{l=0}^{r-1} P_{il}U_{lk} \quad \text{for } 0 \le i < r \text{ and } r \le k \le N. \end{aligned}$$

This equation was derived earlier by first step analysis in (3.26).

Exercises

3.7.1 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.2 & 0.5 & 0.2 \\ 2 & 0.1 & 0.2 & 0.6 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

The transition probability matrix corresponding to the nonabsorbing states is

$$\mathbf{Q} = \begin{array}{c|cc} & 1 & 2 \\ \hline 0 & 0.2 & 0.5 \\ 1 & 0.2 & 0.6 \end{array}.$$

Calculate the matrix inverse to $\mathbf{I} - \mathbf{Q}$, and from this determine
(a) the probability of absorption into state 0 starting from state 1;
(b) the mean time spent in each of states 1 and 2 prior to absorption.

3.7.2 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.3 & 0 & 0.7 & 0 \\ 2 & 0 & 0.3 & 0 & 0.7 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

The transition probability matrix corresponding to the nonabsorbing states is

$$\mathbf{Q} = \begin{array}{c|cc} & 1 & 2 \\ \hline 0 & 0 & 0.7 \\ 1 & 0.3 & 0 \end{array}.$$

Calculate the matrix inverse to $\mathbf{I} - \mathbf{Q}$, and from this determine
(a) the probability of absorption into state 0 starting from state 1;
(b) the mean time spent in each of states 1 and 2 prior to absorption.

Problems

3.7.1 A zero-seeking device operates as follows: If it is in state m at time n, then at time $n+1$ its position is uniformly distributed over the states $0, 1, \ldots, m-1$. State 0 is absorbing. Find the inverse of the $\mathbf{I} - \mathbf{Q}$ matrix for the transient states $1, 2, \ldots, m$.

3.7.2 A zero-seeking device operates as follows: If it is in state j at time n, then at time $n+1$ its position is 0 with probability $1/j$, and its position is k (where k is one of the states $1, 2, \ldots, j-1$) with probability $2k/j^2$. State 0 is absorbing. Find the inverse of the $\mathbf{I} - \mathbf{Q}$ matrix.

3.7.3 Let X_n be an absorbing Markov chain whose transition probability matrix takes the form given in equation (3.76). Let $\mathbf{W}$ be the fundamental matrix, the matrix inverse of $\mathbf{I} - \mathbf{Q}$. Let

$$T = \min\{n \ge 0; r \le n \le N\}$$

be the random time of absorption (recall that states $r, r+1, \ldots, N$ are the absorbing states). Establish the joint distribution

$$\Pr\{X_{T-1} = j, X_T = k | X_0 = i\} = W_{ij}P_{jk} \quad \text{for } 0 \le i,j < r;\, r \le k \le N,$$

whence

$$\Pr\{X_{T-1} = j | X_0 = i\} = \sum_{k=r}^{N} W_{ij}P_{jk} \quad \text{for } 0 \le i,j < r.$$

3.7.4 The possible states for a Markov chain are the integers $0, 1, \ldots, N$, and if the chain is in state j, at the next step it is equally likely to be in any of the states $0, 1, \ldots, j-1$. Formally,

$$P_{ij} = \begin{cases} 1, & \text{if } i = j = 0, \\ 0 & \text{if } 0 < i \le j \le N, \\ 1/i, & \text{if } 0 \le j < i \le N. \end{cases}$$

(a) Determine the fundamental matrix for the transient states $1, 2, \ldots, N$.
(b) Determine the probability distribution for the last positive integer that the chain visits.

3.7.5 *Computer Challenge*. Consider the partial sums:

$$S_0 = k \quad \text{and} \quad S_m = k + \xi_1 + \cdots + \xi_m, \ k > 0,$$

where $\xi_1, \xi_2, \ldots$ are independent and identically distributed as

$$\Pr\{\xi = 0\} = 1 - \frac{2}{\pi}$$

and

$$\Pr\{\xi = \pm j\} = \frac{2}{\pi(4j^2 - 1)}, j = 1, 2, \ldots.$$

Can you find an explicit formula for the mean time v_k for the partial sums starting from $S_0 = k$ to exit the interval $[0, N] = \{0, 1, \ldots, N\}$? In another context, the answer was found by computing it in a variety of special cases.

Note: A simple random walk *on the* integer plane moves according to the rule: If $(X_n, Y_n) = (i, j)$, then the next position is equally likely to be any of the four points $(i+1, j), (i-1, j), (i, j+1)$, or $(i, j-1)$. Let us suppose that the process starts at the point $(X_0, Y_0) = (k, k)$ on the diagonal, and we observe the process only when it visits the diagonal. Formally, we define

$$\tau_1 = \min\{n > 0; X_n = Y_n\},$$

and

$$\tau_m = \min\{n > \tau_{m-1}; X_n = Y_n\}.$$

It is not hard to show that

$$S_0 = k, \ S_m = X_{\tau_m} = Y_{\tau_m}, \ m > 0,$$

is a version of the above partial sum process.

3.8 Branching Processes*

Suppose an organism at the end of its lifetime produces a random number ξ of offspring with probability distribution

$$\Pr\{\xi = k\} = p_k \quad \text{for } k = 0, 1, 2, \ldots, \tag{3.94}$$

* Branching processes are Markov chains of a special type. Sections 3.8 and 3.9 are not prerequisites to the later chapters.

where as usual, $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. We assume that all offspring act independently of each other and at the ends of their lifetimes (for simplicity, the lifespans of all organisms are assumed to be the same) individually have progeny in accordance with the probability distribution (3.94), thus propagating their species. The process $\{X_n\}$, where X_n is the population size at the nth generation, is a Markov chain of special structure called a *branching process*.

The Markov property may be reasoned simply as follows. In the nth generation, the X_n individuals independently give rise to numbers of offspring $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{X_n}^{(n)}$, and hence the cumulative number produced for the $(n+1)$st generation is

$$X_{n+1} = \xi_1^{(n)} + \xi_2^{(n)} + \cdots + \xi_{X_n}^{(n)}. \tag{3.95}$$

3.8.1 Examples of Branching Processes

There are numerous examples of Markov branching processes that arise naturally in various scientific disciplines. We list some of the more prominent cases.

Electron Multipliers

An electron multiplier is a device that amplifies a weak current of electrons. A series of plates are set up in the path of electrons emitted by a source. Each electron, as it strikes the first plate, generates a random number of new electrons, which in turn strike the next plate and produce more electrons, and so forth. Let X_0 be the number of electrons initially emitted and X_1 be the number of electrons produced on the first plate by the impact due to the X_0 initial electrons; in general, let X_n be the number of electrons emitted from the nth plate due to electrons emanating from the $(n-1)$st plate. The sequence of random variables $X_0, X_1, X_2, \ldots, X_n, \ldots$ constitutes a branching process.

Neutron Chain Reaction

A nucleus is split by a chance collision with a neutron. The resulting fission yields a random number of new neutrons. Each of these secondary neutrons may hit some other nucleus, producing a random number of additional neutrons, and so forth. In this case, the initial number of neutrons is $X_0 = 1$. The first generation of neutrons comprises all those produced from the fission caused by the initial neutron. The size of the first generation is a random variable X_1. In general, the population X_n at the nth generation is produced by the chance hits of the X_{n-1} individual neutrons of the $(n-1)$st generation.

Survival of Family Names

The family name is inherited by sons only. Suppose that each individual has probability p_k of having k male offspring. Then, from one individual there result the 1st, 2nd, ..., nth, ... generations of descendants. We may investigate the distribution of such random variables as the number of descendants in the nth generation, or the probability that the family name will eventually become extinct. Such questions will be dealt with beginning in Section 3.8.3.