Let r denote the opportunity cost per unit time of cash on hand. Then, $E[R_i] = rW_s$, while $E[T_i] = v_s$. Since these quantities were determined in (3.57) and (3.59), we have

$$\text{Long run average cost} = \frac{K + (1/3)rs\left(\mathcal{S}^2 - s^2\right)}{s(\mathcal{S} - s)}. \tag{3.60}$$

In order to use calculus to determine the cost-minimizing values for $\mathcal{S}$ and s, it simplifies matters if we introduce the new variable $x = s/\mathcal{S}$. Then, (3.60) becomes

$$\text{Long run average cost} = \frac{K + (1/3)r\mathcal{S}^3 x\left(1 - x^2\right)}{\mathcal{S}^2 x(1 - x)},$$

whence

$$\frac{\mathrm{d(average\ cost)}}{\mathrm{d}x} = 0 = -\frac{K(1 - 2x)}{\mathcal{S}^2 x^2 (1 - x)^2} + \frac{1}{3} r\mathcal{S},$$

$$\frac{\mathrm{d(average\ cost)}}{\mathrm{d}\mathcal{S}} = 0 = -\frac{2K}{\mathcal{S}^3 x(1 - x)} + \frac{r(1 + x)}{3},$$

which yield

$$x_{\mathrm{opt}} = \frac{1}{3} \quad \text{and} \quad \mathcal{S}_{\mathrm{opt}} = 3s_{\mathrm{opt}} = 3\sqrt[3]{\frac{3K}{4r}}.$$

Implementing the cash management strategy with the values s_{opt} and $\mathcal{S}_{\mathrm{opt}}$ results in the optimal balance between transaction costs and the opportunity cost of holding cash on hand.

3.6.3 The Success Runs Markov Chain

Consider the success runs Markov chain on $N + 1$ states whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & N \end{array} \\ \left\| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ p_1 & r_1 & q_1 & 0 & \cdots & 0 \\ p_2 & 0 & r_2 & q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ p_{N-1} & 0 & 0 & 0 & \cdots & q_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{array} \right\| \end{array}.$$

Note that states 0 and N are absorbing; once the process reaches one of these two states it remains there.

Let T be the hitting time to states 0 or N,

$$T = \min\{n \geq 0; X_n = 0 \text{ or } X_n = N\}.$$

Exercises

3.6.1 A rat is put into the linear maze as shown:

0 shock	1	2	3	4	5 food

(a) Assume that the rat is equally likely to move right or left at each step. What is the probability that the rat finds the food before getting shocked?

(b) As a result of learning, at each step the rat moves to the right with probability $p > \frac{1}{2}$ and to the left with probability $q = 1 - p < \frac{1}{2}$. What is the probability that the rat finds the food before getting shocked?

3.6.2 Customer accounts receivable at Smith Company are classified each month according to

0: Current
1: 30–60 days past due
2: 60–90 days past due
3: Over 90 days past due

Consider a particular customer account and suppose that it evolves month to month as a Markov chain $\{X_n\}$ whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.9 & 0.1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.3 & 0 & 0 & 0.7 \\ 0.2 & 0 & 0 & 0.8 \end{array} \right\| \end{array}.$$

Suppose that a certain customer's account is now in state 1: 30–60 days past due. What is the probability that this account will be paid (and thereby enter state 0: Current) before it becomes over 90 days past due? That is, let $T = \min\{n \geq 0; X_n = 0 \text{ or } X_n = 3\}$. Determine $\Pr\{X_T = 0 | X_0 = 1\}$.

3.6.3 Players A and B each have \$50 at the beginning of a game in which each player bets \$1 at each play, and the game continues until one player is broke. Suppose there is a constant probability $p = 0.492929\ldots$ that Player A wins on any given bet. What is the mean duration of the game?

3.6.4 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time until absorption. Do this first using the basic first step approach of equation (3.24), and second using the particular formula for v_i that follows equation (3.54), which applies for a random walk in which $p \neq q$.

Problems

3.6.1 The probability of gambler's ruin

$$u_i = \Pr\{X_T = 0 | X_0 = i\} \tag{3.61}$$

satisfies the first step analysis equation

$$u_i = q_i u_{i-1} + r_i u_i + p_i u_{i+1} \quad \text{for } i = 1, \ldots, N-1,$$

and

$$u_0 = 1, \quad u_N = 0.$$

The solution is

$$u_i = \frac{\rho_i + \cdots + \rho_{N-1}}{1 + \rho_1 + \rho_2 + \cdots + \rho_{N-1}}, \quad i = 1, \ldots, N-1, \tag{3.62}$$

where

$$\rho_k = \frac{q_1 q_2 \cdots q_k}{p_1 p_2 \cdots p_k}, \quad k = 1, \ldots, N-1. \tag{3.63}$$

3.6.2 The mean hitting time

$$v_k = E[T | X_0 = k] \tag{3.64}$$

satisfies the equations

$$v_k = 1 + q_k v_{k-1} + r_k v_k + p_k v_{k+1} \quad \text{and} \quad v_0 = v_N = 0. \tag{3.65}$$

The solution is

$$v_k = \left(\frac{\Phi_1 + \cdots + \Phi_{N-1}}{1 + \rho_1 + \cdots + \rho_{N-1}}\right)(1 + \rho_1 + \cdots + \rho_{k-1}) - (\Phi_1 + \cdots + \Phi_{k-1}) \quad \text{for } k = 1, \ldots, N-1, \tag{3.66}$$

where ρ_i is given in (3.63) and

$$\Phi_i = \left(\frac{1}{q_1} + \frac{1}{q_2\rho_1} + \cdots + \frac{1}{q_i\rho_{i-1}}\right)\rho_i = \frac{q_2 \cdots q_i}{p_1 \cdots p_i} + \frac{q_3 \cdots q_i}{p_2 \cdots p_i} + \cdots + \frac{q_i}{p_{i-1}p_i} + \frac{1}{p_i} \quad \text{for } i = 1, \ldots, N-1. \tag{3.67}$$

3.6.3 Fix a state k, where $0 < k < N$, and let W_{ik} be the mean total visits to state k starting from i. Formally, the definition is

$$W_{ik} = E\left[\sum_{n=0}^{T-1} \mathbf{1}\{X_n = k\} | X_0 = i\right], \tag{3.68}$$

where

$$\mathbf{1}\{X_n = k\} = \begin{cases} 1 & \text{if } X_n = k, \\ 0 & \text{if } X_n \neq k. \end{cases}$$

Then, W_{ik} satisfies the equations

$$W_{ik} = \delta_{ik} + q_i W_{i-1,k} + r_i W_{ik} + p_i W_{i+1,k} \quad \text{for } i = 1, \ldots, N-1$$

and

$$W_{0k} = W_{Nk} = 0,$$

where

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

The solution is

$$W_{ik} = \begin{cases} \dfrac{(1 + \cdots + \rho_{i-1})(\rho_k + \cdots + \rho_{N-1})}{1 + \cdots + \rho_{N-1}}\left(\dfrac{1}{q_k\rho_{k-1}}\right) & \text{for } i \leq k, \\ \left[\dfrac{(1 + \cdots + \rho_{i-1})(\rho_k + \cdots + \rho_{N-1})}{1 + \cdots + \rho_{N-1}} - (\rho_k + \cdots + \rho_{i-1})\right]\left(\dfrac{1}{q_k\rho_{k-1}}\right) & \text{for } i \geq k. \end{cases} \tag{3.69}$$

3.6.4 The probability of absorption at 0 starting from state k

$$u_k = \Pr\{X_T = 0 | X_0 = k\} \tag{3.70}$$

satisfies the equation

$$u_k = p_k + r_k u_k + q_k u_{k+1},$$
for $k = 1, \ldots, N-1$ and $u_0 = 1, u_N = 0$.

The solution is

$$u_k = 1 - \left(\frac{q_k}{p_k+q_k}\right)\cdots\left(\frac{q_{N-1}}{p_{N-1}+q_{N-1}}\right) \quad \text{for } k = 1,\ldots,N-1. \tag{3.71}$$

3.6.5 The mean hitting time

$$v_k = E[T|X_0 = k] \tag{3.72}$$

satisfies the equation

$$v_k = 1 + r_k v_k + q_k v_{k+1} \quad \text{for } k = 1,\ldots,N-1 \quad \text{and} \quad v_0 = v_N = 0.$$

The solution is

$$v_k = \frac{1}{p_k+q_k} + \frac{\pi_{k,k+1}}{p_{k+1}+q_{k+1}} + \cdots + \frac{\pi_{k,N-1}}{p_{N-1}+q_{N-1}}, \tag{3.73}$$

where

$$\pi_{kj} = \left(\frac{q_k}{p_k+q_k}\right)\left(\frac{q_{k+1}}{p_{k+1}+q_{k+1}}\right)\cdots\left(\frac{q_{j-1}}{p_{j-1}+q_{j-1}}\right) \quad \text{for } k < j. \tag{3.74}$$

3.6.6 Fix a state $j (0 < j < N)$ and let W_{ij} be the mean total visits to state j starting from state i [see equation (3.68)]. Then,

$$W_{ij} = \begin{cases} \dfrac{1}{p_i+q_i} & \text{for } j = i, \\ \left(\dfrac{q_i}{p_i+q_i}\right)\cdots\left(\dfrac{q_{j-1}}{p_{j-1}+q_{j-1}}\right)\dfrac{1}{p_j+q_j} & \text{for } i < j, \\ 0 & \text{for } i > j. \end{cases} \tag{3.75}$$

3.6.7 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time until absorption. Do this first using the basic first step approach of equation (3.24) and second using the particular results for a random walk given in equation (3.66).

3.6.8 Consider the Markov chain $\{X_n\}$ whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} \alpha & 0 & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta = 1$. Determine the mean time to reach state 3 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = \min\{n \geq 0; X_n = 3\}$.

3.6.9 *Computer Challenge*. You have two urns: A and B, with a balls in A and b balls in B. You pick an urn at random, each urn being equally likely, and move a ball from it to the other urn. You do this repeatedly. The game ends when either of the urns becomes empty. The number of balls in A at the nth move is a simple random walk, and the expected duration of the game is $E[T] = ab$ [see equation (3.52)]. Now consider three urns, A, B, and C, with a, b, and c balls, respectively. You pick an urn at random, each being equally likely, and move a ball from it to one of the other two urns, each being equally likely. The game ends when one of the three urns becomes empty. What is the mean duration of the game? If you can guess the general form of this mean time by computing it in a variety of particular cases, it is not particularly difficult to verify it by a first step analysis. What about four urns?

3.7 Another Look at First Step Analysis*

In this section, we provide an alternative approach to evaluating the functionals treated in Section 3.4. The nth power of a transition probability matrix having both transient and absorbing states is directly evaluated. From these nth powers, it is possible to extract the mean number of visits to a transient state j prior to absorption, the mean time until absorption, and the probability of absorption in any particular absorbing state k. These functionals all depend on the initial state $X_0 = i$, and as a by-product of the derivation, we show that, as functions of this initial state i, these functionals satisfy their appropriate first step analysis equations.

Consider a Markov chain whose states are labeled $0, 1, \ldots, N$. States $0, 1, \ldots, r-1$ are transient in that $P_{ij}^{(n)} \to 0$ as $n \to \infty$ for $0 \leq i, j < r$, while states $r, \ldots, N$ are absorbing, or trap, and here $P_{ii} = 1$ for $r \leq i \leq N$. The transition matrix has the form

$$\mathbf{P} = \left\| \begin{array}{cc} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{array} \right\|, \tag{3.76}$$

* This section contains material at a more difficult level. It is not prerequisite to what follows.