From the theory of matrices, we recognize the relation (3.11) as the formula for matrix multiplication so that $\mathbf{P}^{(n)} = \mathbf{P} \times \mathbf{P}^{(n-1)}$. By iterating this formula, we obtain

$$\mathbf{P}^{(n)} = \underbrace{\mathbf{P} \times \mathbf{P} \times \cdots \times \mathbf{P}}_{n \text{ factors}} = \mathbf{P}^n; \tag{3.12}$$

in other words, the n-step transition probabilities $P_{ij}^{(n)}$ are the entries in the matrix $\mathbf{P}^n$, the nth power of $\mathbf{P}$.

Proof. The proof proceeds via a *first step analysis*, a breaking down, or analysis, of the possible transitions on the first step, followed by an application of the Markov property. The event of going from state i to state j in n transitions can be realized in the mutually exclusive ways of going to some intermediate state $k (k = 0, 1, \ldots)$ in the first transition, and then going from state k to state j in the remaining $(n-1)$ transitions. Because of the Markov property, the probability of the second transition is $P_{kj}^{(n-1)}$ and that of the first is clearly P_{ik}. If we use the law of total probability, then (3.11) follows. The steps are

$$\begin{aligned} P_{ij}^{(n)} &= \Pr\{X_n = j | X_0 = i\} = \sum_{k=0}^{\infty} \Pr\{X_n = j, X_1 = k | X_0 = i\} \\ &= \sum_{k=0}^{\infty} \Pr\{X_1 = k | X_0 = i\} \Pr\{X_n = j | X_0 = i, X_1 = k\} \\ &= \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}. \end{aligned}$$

If the probability of the process initially being in state j is p_j, i.e., the distribution law of X_0 is $\Pr\{X_0 = j\} = p_j$, then the probability of the process being in state k at time n is

$$p_k^{(n)} = \sum_{j=0}^{\infty} p_j P_{jk}^{(n)} = \Pr\{X_n = k\}. \tag{3.13}$$

■

Exercises

3.2.1 A Markov chain $\{X_n\}$ on the states $0, 1, 2$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{array} \right\| \end{array}.$$

(a) Compute the two-step transition matrix P^2.
(b) What is $\Pr\{X_3 = 1 | X_1 = 0\}$?
(c) What is $\Pr\{X_3 = 1 | X_0 = 0\}$?

3.2.2 A particle moves among the states $0, 1, 2$ according to a Markov process whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\| \end{array}.$$

Let X_n denote the position of the particle at the nth move. Calculate $\Pr\{X_n = 0 | X_0 = 0\}$ for $n = 0, 1, 2, 3, 4$.

3.2.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_4 = 1 | X_0 = 0\}.$$

3.2.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_2 = 2\}$.

3.2.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1 | X_0 = 0\}.$$

3.2.6 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

Problems

3.2.1 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{array} \right\| \end{array}.$$

Suppose that the initial distribution is $p_i = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that $\Pr\{X_n = k\} = \frac{1}{4}, k = 0, 1, 2, 3$, for all n. Can you deduce a general result from this example?

3.2.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Let X_0 be the signal that is sent, and let X_n be the signal that is received at the nth stage. Suppose X_n is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha, (0 < \alpha < 1)$. Determine $\Pr\{X_5 = 0 | X_0 = 0\}$, the probability of correct transmission through five stages.

3.2.3 Let X_n denote the quality of the nth item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective." Suppose that X_n evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}.$$

What is the probability that the fourth item is defective given that the first item is defective?

3.2.4 Suppose X_n is a two-state Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} \alpha & 1-\alpha \\ 1-\beta & \beta \end{array} \right\| \end{array}.$$

Then, $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0,0), (0,1), (1,0)$, and $(1,1)$. Determine the transition probability matrix.

3.2.5 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

The Markov chain starts at time zero in state $X_0 = 0$. Let

$$T = \min\{n \geq 0; X_n = 2\}$$

be the first time that the process reaches state 2. Eventually, the process will reach and be absorbed into state 2. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $\Pr\{X_3 = 0 | X_0, T > 3\}$.

Hint: The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.

3.3 Some Markov Chain Models

Markov chains can be used to model and quantify a large number of natural physical, biological, and economic phenomena that can be described by them. This is enhanced by the amenability of Markov chains to quantitative manipulation. In this section, we give several examples of Markov chain models that arise in various parts of science. General methods for computing certain functionals on Markov chains are derived in the following section.

3.3.1 *An Inventory Model*

Consider a situation in which a commodity is stocked in order to satisfy a continuing demand. We assume that the replenishment of stock takes place at the end of periods labeled $n = 0, 1, 2, \ldots$, and we assume that the total aggregate demand for the commodity during period n is a random variable ξ_n whose distribution function is independent of the time period,

$$\Pr\{\xi_n = k\} = a_k \quad \text{for } k = 0, 1, 2, \ldots, \tag{3.14}$$

will serve to illustrate the concepts and approach. We introduce the states

E_0: Prepuberty, E_3: Divorced,
E_1: Single, E_4: Widowed,
E_2: Married, E_5: Δ,

and we are interested in the mean duration spent in state E_2: Married, since this corresponds to the state of maximum fecundity. To illustrate the computations, we will suppose the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \left\| \begin{array}{cccccc} 0 & 0.9 & 0 & 0 & 0 & 0.1 \\ 0 & 0.5 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.4 & 0.5 & 0 & 0.1 \\ 0 & 0 & 0.4 & 0 & 0.5 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{array} \right\| \end{array}.$$

In practice, such a matrix would be estimated from demographic data.

Every person begins in state E_0 and ends in state E_5, but a variety of intervening states may be visited. We wish to determine the mean duration spent in state E_2: Married. The powerful approach of *first step analysis* begins by considering the slightly more general problem in which the initial state is varied. Let $w_i = W_{i2}$ be the mean duration in state E_2 given the initial state $X_0 = E_i$ for $i = 0, 1, \ldots, 5$. We are interested in w_0, the mean duration corresponding to the initial state E_0.

First step analysis breaks down, or analyzes, the possibilities arising in the first transition, and using the Markov property, an equation that relates $w_0, \ldots, w_5$ results.

We begin by considering w_0. From state E_0, a transition to one of the states E_1 or E_5 occurs, and the mean duration spent in E_2 starting from E_0 must be the appropriately weighted average of w_1 and w_5. That is,

$$w_0 = 0.9w_1 + 0.1w_5.$$

Proceeding in a similar manner, we obtain

$$w_1 = 0.5w_1 + 0.4w_2 + 0.1w_5.$$

The situation changes when the process begins in state E_2 because in counting the mean duration spent in E_2, we must count this initial visit plus any subsequent visits that may occur. Thus, for E_2, we have

$$w_2 = 1 + 0.6w_2 + 0.2w_3 + 0.1w_4 + 0.1w_5.$$

The other states give us

$$\begin{aligned} w_3 &= 0.4w_2 + 0.5w_3 + 0.1w_5, \\ w_4 &= 0.4w_2 + 0.5w_4 + 0.1w_5, \\ w_5 &= w_5. \end{aligned}$$

Since state E_5 corresponds to death, it is clear that we must have $w_5 = 0$. With this prescription, the reduced equations become, after elementary simplification,

$$\begin{aligned} -1.0w_0 + 0.9w_1 \qquad\qquad &= 0, \\ -0.5w_1 + 0.4w_2 \qquad &= 0, \\ -0.4w_2 + 0.2w_3 + 0.1w_4 &= -1, \\ 0.4w_2 - 0.5w_3 \qquad &= 0, \\ 0.4w_2 \qquad - 0.5w_4 &= 0. \end{aligned}$$

The unique solution is

$$w_0 = 4.5, \quad w_1 = 5.00, \quad w_2 = 6.25, \quad w_3 = w_4 = 5.00.$$

Each female, on the average, spends $w_0 = W_{02} = 4.5$ periods in the childbearing state E_2 during her lifetime.

Exercises

3.4.1 Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

3.4.2 Consider the Markov chain whose transition probablity matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.
(b) Determine the mean time to absorption.

3.4.3 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

3.4.4 A coin is tossed repeatedly until two successive heads appear. Find the mean number of tosses required.

Hint: Let X_n be the cumulative number of successive heads. The state space is $0, 1, 2$, and the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Determine the mean time to reach state 2 starting from state 0 by invoking a first step analysis.

3.4.5 A coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose the first coin toss results in a head. Find the probability that the game ends with two successive tails.

3.4.6 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

3.4.7 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

3.4.8 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

3.4.9 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the probability that the process is absorbed into state 0. Compare this with the $(1,0)$th entry in the matrix powers $\mathbf{P}^2, \mathbf{P}^4, \mathbf{P}^8$, and $\mathbf{P}^{16}$.

Problems

3.4.1 Which will take fewer flips, on average: successively flipping a quarter until the pattern *HHT* appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern *HTH* appears? Can you explain why these are different?

3.4.2 A zero-seeking device operates as follows: If it is in state m at time n, then at time $n+1$, its position is uniformly distributed over the states $0, 1, \ldots, m-1$. Find the expected time until the device first hits zero starting from state m.

Note: This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.

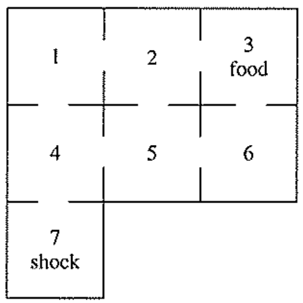
3.4.3 A zero-seeking device operates as follows: If it is in state j at time n, then at time $n+1$, its position is 0 with probability $1/j$, and its position is k (where k is one of the states $1, 2, \ldots, j-1$) with probability $2k/j^2$. Find the expected time until the device first hits zero starting from state m.

3.4.4 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.2 & 0.5 & 0.2 \\ 2 & 0.1 & 0.2 & 0.6 & 0.1 \\ 3 & 0.2 & 0.2 & 0.3 & 0.3 \end{array}.$$

Starting in state $X_0 = 1$, determine the probability that the process never visits state 2. Justify your answer.

3.4.5 A white rat is put into compartment 4 of the maze shown here:



It moves through the compartments at random; i.e., if there are k ways to leave a compartment, it chooses each of these with probability $1/k$. What is the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7?

3.4.6 Consider the Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & q & p & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 \\ 2 & q & 0 & 0 & p & 0 \\ 3 & q & 0 & 0 & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array},$$

where $p + q = 1$. Determine the mean time to reach state 4 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = \min\{n \geq 0; X_n = 4\}$.

Hint: Let $v_i = E[T|X_0 = i]$ for $i = 0, 1, \ldots, 4$. Establish equations for $v_0, v_1, \ldots, v_4$ by using a first step analysis and the boundary condition $v_4 = 0$. Then, solve for v_0.

3.4.7 Let X_n be a Markov chain with transition probabilities P_{ij}. We are given a "discount factor" β with $0 < \beta < 1$ and a cost function $c(i)$, and we wish to determine the total expected discounted cost starting from state i, defined by

$$h_i = E\left[\sum_{n=0}^{\infty} \beta^n c(X_n) | X_0 = i\right].$$

Using a first step analysis show that h_i satisfies the system of linear equations

$$h_i = c(i) + \beta \sum_j P_{ij} h_j \quad \text{for all states } i.$$

3.4.8 An urn contains five red and three green balls. The balls are chosen at random, one by one, from the urn. If a red ball is chosen, it is removed. Any green ball that is chosen is returned to the urn. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

3.4.9 An urn contains five red and three yellow balls. The balls are chosen at random, one by one, from the urn. Each ball removed is replaced in the urn by a yellow ball. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

3.4.10 You have five fair coins. You toss them all so that they randomly fall heads or tails. Those that fall tails in the first toss you pick up and toss again. You toss again those that show tails after the second toss, and so on, until all show heads. Let X be the number of coins involved in the *last* toss. Find $\Pr\{X = 1\}$.

3.4.11 An urn contains two red and two green balls. The balls are chosen at random, one by one, and removed from the urn. The selection process continues until all of the green balls have been removed from the urn. What is the probability that a single red ball is in the urn at the time that the last green ball is chosen?

3.4.12 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.3 & 0.2 & 0.5 \\ 1 & 0.5 & 0.1 & 0.4 \\ 2 & 0 & 0 & 1 \end{array}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that when the process moves into state 2, it does so from state 1?

Hint: Let $T = \min\{n \geq 0; X_n = 2\}$, and let

$$z_i = \Pr\{X_{T-1} = 1 | X_0 = i\} \quad \text{for } i = 0, 1.$$

Establish and solve the first step equations

$$\begin{aligned} z_0 &= \quad 0.3z_0 + 0.2z_1, \\ z_1 &= 0.4 + 0.5z_0 + 0.1z_1. \end{aligned}$$

3.4.13 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that the time $T = \min\{n \geq 0; X_n = 2\}$ is an odd number?

3.4.14 A single die is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 5 or 7. What is the probability that the game stops at a sum of 5?

3.4.15 A simplified model for the spread of a rumor goes this way: There are $N = 5$ people in a group of friends, of which some have heard the rumor and the others have not. During any single period of time, two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friends is just as likely as between any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n denote the number of friends who have heard the rumor at the end of the nth period.

Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

3.4.16 An urn contains five tags, of which three are red and two are green. A tag is randomly selected from the urn and replaced with a tag of the opposite color. This continues until only tags of a single color remain in the urn. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 3$. What is the probability that the game ends with the urn containing only red tags?

3.4.17 The *damage* X_n of a system subjected to wear is a Markov chain with the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

The system starts in state 0 and fails when it first reaches state 2. Let $T = \min\{n \geq 0; X_n = 2\}$ be the time of failure. Use a first step analysis to evaluate $\phi(s) = E\left[s^T\right]$ for a fixed number $0 < s < 1$. (This is called the *generating function* of T. See Section 3.9.)

3.4.18 *Time-dependent transition probabilities.* A well-disciplined man, who smokes exactly one half of a cigar each day, buys a box containing N cigars. He cuts a cigar in half, smokes half, and returns the other half to the box. In general, on a day in which his cigar box contains w whole cigars and h half cigars, he will pick one of the $w + h$ smokes at random, each whole and half cigar being equally likely, and if it is a half cigar, he smokes it. If it is a whole cigar, he cuts it in half, smokes one piece, and returns the other to the box. What is the expected value of T, the day on which the last whole cigar is selected from the box?

Hint: Let X_n be the number of whole cigars in the box after the nth smoke. Then, X_n is a Markov chain whose transition probabilities vary with n. Define $v_n(w) = E[T | X_n = w]$. Use a first step analysis to develop a recursion for $v_n(w)$ and show that the solution is

$$v_n(w) = \frac{2Nw + n + 2w}{w + 1} - \sum_{k=1}^{w} \frac{1}{k},$$

whence

$$E[T] = v_0(N) = 2N - \sum_{k=1}^{N} \frac{1}{k}.$$

3.4.19 *Computer Challenge.* Let N be a positive integer and let $Z_1, \ldots, Z_N$ be independent random variables, each having the geometric distribution

$$\Pr\{Z = k\} = \left(\frac{1}{2}\right)^k, \quad \text{for } k = 1, 2, \ldots.$$

Since these are discrete random variables, the maximum among them may be unique, or there may be ties for the maximum. Let p_N be the probability that the maximum is unique. How does p_N behave when N is large? (Alternative formulation: You toss N dimes. Those that are heads you set aside; those that are tails you toss again. You repeat this until all of the coins are heads. Then, p_N is the probability that the last toss was of a single coin.)

3.5 Some Special Markov Chains

We introduce several particular Markov chains that arise in a variety of applications.