This shows that all finite-dimensional probabilities are specified once the transition probabilities and initial distribution are given, and in this sense, the process is defined by these quantities.

Related computations show that (3.1) is equivalent to the Markov property in the form

$$\begin{aligned}&\Pr\{X_{n+1}=j_1,\ldots,X_{n+m}=j_m|X_0=i_0,\ldots,X_n=i_n\}\\&\quad=\Pr\{X_{n+1}=j_1,\ldots,X_{n+m}=j_m|X_n=i_n\}\end{aligned}\tag{3.9}$$

for all time points n, m and all states $i_0,\ldots,i_n,j_1,\ldots,j_m$. In other words, once (3.9) is established for the value $m=1$, it holds for all $m \geq 1$ as well.

Exercises

3.1.1 A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\mathbf{P}=\begin{array}{c}\\0\\1\\2\end{array}\begin{array}{c}\begin{array}{ccc}0&1&2\end{array}\\\left\|\begin{array}{ccc}0.1&0.2&0.7\\0.9&0.1&0\\0.1&0.8&0.1\end{array}\right\|\end{array}$$

and initial distribution $p_0=\Pr\{X_0=0\}=0.3, p_1=\Pr\{X_0=1\}=0.4$, and $p_2=\Pr\{X_0=2\}=0.3$. Determine $\Pr\{X_0=0,X_1=1,X_2=2\}$.

3.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}=\begin{array}{c}\\0\\1\\2\end{array}\begin{array}{c}\begin{array}{ccc}0&1&2\end{array}\\\left\|\begin{array}{ccc}0.7&0.2&0.1\\0&0.6&0.4\\0.5&0&0.5\end{array}\right\|\end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_2=1,X_3=1|X_1=0\}\quad\text{and}\quad\Pr\{X_1=1,X_2=1|X_0=0\}.$$

3.1.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}=\begin{array}{c}\\0\\1\\2\end{array}\begin{array}{c}\begin{array}{ccc}0&1&2\end{array}\\\left\|\begin{array}{ccc}0.6&0.3&0.1\\0.3&0.3&0.4\\0.4&0.1&0.5\end{array}\right\|\end{array}.$$

If it is known that the process starts in state $X_0=1$, determine the probability $\Pr\{X_0=1,X_1=0,X_2=2\}$.

3.1.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\}.$$

3.1.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities

$$\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}.$$

Problems

3.1.1 A simplified model for the spread of a disease goes this way: The total population size is $N = 5$, of which some are diseased and the remainder are healthy. During any single period of time, two people are selected at random from the population and assumed to interact. The selection is such that an encounter between any pair of individuals in the population is just as likely as between any other pair. If one of these persons is diseased and the other not, with probability $\alpha = 0.1$ the disease is transmitted to the healthy person. Otherwise, no disease transmission takes place. Let X_n denote the number of diseased persons in the population at the end of the nth period. Specify the transition probability matrix.

3.1.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(a) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(b) Determine the probability that a correct signal is received at stage 2.

Hint: This is $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$.

3.1.3 Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1-\alpha$. Similarly, a defective item is followed by another defective item with probability β and is followed by a good item with probability $1-\beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth item?

3.1.4 The random variables $\xi_1, \xi_2, \ldots$ are independent and with the common probability mass function

$k=$	0	1	2	3
$\Pr\{\xi=k\}=$	0.1	0.3	0.2	0.4

Set $X_0=0$, and let $X_n=\max\{\xi_1,\ldots,\xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

3.2 Transition Probability Matrices of a Markov Chain

A Markov chain is completely defined by its one-step transition probability matrix and the specification of a probability distribution on the state of the process at time 0. The analysis of a Markov chain concerns mainly the calculation of the probabilities of the possible realizations of the process.

Central in these calculations are the n-step transition probability matrices $\mathbf{P}^{(n)} = \|P_{ij}^{(n)}\|$. Here, $P_{ij}^{(n)}$ denotes the probability that the process goes from state i to state j in n transitions. Formally,

$$P_{ij}^{(n)} = \Pr\{X_{m+n}=j|X_m=i\}. \tag{3.10}$$

Observe that we are dealing only with temporally homogeneous processes having stationary transition probabilities, since otherwise the left side of (3.10) would also depend on m.

The Markov property allows us to express (3.10) in terms of $\|P_{ij}\|$ as stated in the following theorem.

Theorem 3.1. *The n-step transition probabilities of a Markov chain satisfy*

$$P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik}P_{kj}^{(n-1)}, \tag{3.11}$$

where we define

$$P_{ij}^{(0)} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

From the theory of matrices, we recognize the relation (3.11) as the formula for matrix multiplication so that $\mathbf{P}^{(n)} = \mathbf{P} \times \mathbf{P}^{(n-1)}$. By iterating this formula, we obtain

$$\mathbf{P}^{(n)} = \underbrace{\mathbf{P} \times \mathbf{P} \times \cdots \times \mathbf{P}}_{n \text{ factors}} = \mathbf{P}^n; \tag{3.12}$$

in other words, the n-step transition probabilities $P_{ij}^{(n)}$ are the entries in the matrix $\mathbf{P}^n$, the nth power of $\mathbf{P}$.

Proof. The proof proceeds via a *first step analysis*, a breaking down, or analysis, of the possible transitions on the first step, followed by an application of the Markov property. The event of going from state i to state j in n transitions can be realized in the mutually exclusive ways of going to some intermediate state $k (k = 0, 1, \ldots)$ in the first transition, and then going from state k to state j in the remaining $(n-1)$ transitions. Because of the Markov property, the probability of the second transition is $P_{kj}^{(n-1)}$ and that of the first is clearly P_{ik}. If we use the law of total probability, then (3.11) follows. The steps are

$$\begin{aligned} P_{ij}^{(n)} &= \Pr\{X_n = j | X_0 = i\} = \sum_{k=0}^{\infty} \Pr\{X_n = j, X_1 = k | X_0 = i\} \\ &= \sum_{k=0}^{\infty} \Pr\{X_1 = k | X_0 = i\} \Pr\{X_n = j | X_0 = i, X_1 = k\} \\ &= \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}. \end{aligned}$$

If the probability of the process initially being in state j is p_j, i.e., the distribution law of X_0 is $\Pr\{X_0 = j\} = p_j$, then the probability of the process being in state k at time n is

$$p_k^{(n)} = \sum_{j=0}^{\infty} p_j P_{jk}^{(n)} = \Pr\{X_n = k\}. \tag{3.13}$$

∎

Exercises

3.2.1 A Markov chain $\{X_n\}$ on the states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{array} \right\| \end{array}.$$

the relation (3.11) as the formula for
By iterating this formula, we obtain

(3.12)

ies $P_{ij}^{(n)}$ are the entries in the matrix $\mathbf{P}^n$,

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termediate state $k(k = 0, 1, \ldots)$ in the first
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ility of the second transition is $P_{kj}^{(n-1)}$ and
aw of total probability, then (3.11) follows.

$= j, X_1 = k | X_0 = i\}$

$= j | X_0 = i, X_1 = k\}$

ally being in state j is p_j, i.e., the distribution
probability of the process being in state k at

(3.13)

■

tates 0, 1, 2 has the transition probability matrix

(a) Compute the two-step transition matrix P^2.
(b) What is $\Pr\{X_3 = 1 | X_1 = 0\}$?
(c) What is $\Pr\{X_3 = 1 | X_0 = 0\}$?

3.2.2 A particle moves among the states 0, 1, 2 according to a Markov process whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\| \end{array}.$$

Let X_n denote the position of the particle at the nth move. Calculate $\Pr\{X_n = 0 | X_0 = 0\}$ for $n = 0, 1, 2, 3, 4$.

3.2.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_4 = 1 | X_0 = 0\}.$$

3.2.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_2 = 2\}$.

3.2.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1|X_1 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1|X_0 = 0\}.$$

3.2.6 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array}\right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

Problems

3.2.1 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{array}\right\| \end{array}.$$

Suppose that the initial distribution is $p_i = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that $\Pr\{X_n = k\} = \frac{1}{4}, k = 0, 1, 2, 3$, for all n. Can you deduce a general result from this example?

3.2.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Let X_0 be the signal that is sent, and let X_n be the signal that is received at the nth stage. Suppose X_n is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, $(0 < \alpha < 1)$. Determine $\Pr\{X_5 = 0|X_0 = 0\}$, the probability of correct transmission through five stages.

3.2.3 Let X_n denote the quality of the nth item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective." Suppose that X_n evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\|\begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array}\right\| \end{array}.$$

What is the probability that the fourth item is defective given that the first item is defective?

$1|X_0 = 0\}$.

ansition probability matrix

$p_1 = 0.5$. Determine the probabilities

nsition probability matrix is given by

on is $p_i = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that
ll n. Can you deduce a general result from

a binary message, 0 or 1, through a signal
es, where transmission through each stage is
rror α. Let X_0 be the signal that is sent, and
ed at the nth stage. Suppose X_n is a Markov
; $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, $(0 <$
$= 0\}$, the probability of correct transmission

e nth item produced by a production system
d $X_n = 1$ meaning "defective." Suppose that
hose transition probability matrix is

fourth item is defective given that the first item

3.2.4 Suppose X_n is a two-state Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} \alpha & 1-\alpha \\ 1-\beta & \beta \end{array} \right\| \end{array}.$$

Then, $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Determine the transition probability matrix.

3.2.5 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

The Markov chain starts at time zero in state $X_0 = 0$. Let

$$T = \min\{n \geq 0; X_n = 2\}$$

be the first time that the process reaches state 2. Eventually, the process will reach and be absorbed into state 2. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $\Pr\{X_3 = 0 | X_0, T > 3\}$.

Hint: The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.

3.3 Some Markov Chain Models

Markov chains can be used to model and quantify a large number of natural physical, biological, and economic phenomena that can be described by them. This is enhanced by the amenability of Markov chains to quantitative manipulation. In this section, we give several examples of Markov chain models that arise in various parts of science. General methods for computing certain functionals on Markov chains are derived in the following section.

3.3.1 *An Inventory Model*

Consider a situation in which a commodity is stocked in order to satisfy a continuing demand. We assume that the replenishment of stock takes place at the end of periods labeled $n = 0, 1, 2, \ldots$, and we assume that the total aggregate demand for the commodity during period n is a random variable ξ_n whose distribution function is independent of the time period,

$$\Pr\{\xi_n = k\} = a_k \quad \text{for } k = 0, 1, 2, \ldots, \tag{3.14}$$

a waiting line. During each period
t at least one customer is present.
riod no service is performed. (We
ives at fixed time intervals to give
departs.) During a service period,
tual number of customers that arrive
se distribution is independent of the

$\Pr\{\xi_n = k\} = a_k,$

nt random variables. The state of the
e the number of customers waiting in
r the lapse of one period the state is

(3.18)

ng arrived in this period while a single
variables of the process, we can express

he transition probability matrix may be

l number of new customers, $\sum_{k=0}^{\infty} ka_k$, who
e, then with the passage of time the length
t. On the other hand, if $\sum_{k=0}^{\infty} ka_k < 1$, then
a statistical equilibrium that is described by

for $k = 0, 1, \ldots,$

s to be determined by this model include the
facility is idle, given by π_0, and the long run
system, given by $\sum_{k=0}^{\infty}(1+k)\pi_k$.

Exercises

3.3.1 Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with

$$\Pr\{\xi_n = 0\} = 0.4, \quad \Pr\{\xi_n = 1\} = 0.3, \quad \Pr\{\xi_n = 2\} = 0.3,$$

and suppose $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end-of-period n.

3.3.2 Consider two urns A and B containing a total of N balls. An experiment is performed in which a ball is selected at random (all selections equally likely) at time $t(t = 1, 2, \ldots)$ from among the totality of N balls. Then, an urn is selected at random (A is chosen with probability p and B is chosen with probability q) and the ball previously drawn is placed in this urn. The state of the system at each trial is represented by the number of balls in A. Determine the transition matrix for this Markov chain.

3.3.3 Consider the inventory model of Section 3.3.1. Suppose that $S = 3$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n.

3.3.4 Consider the inventory model of Section 3.3.1. Suppose that $S = 3$ and that the probability distribution for demand is $\Pr\{\xi = 0\} = 0.1$, $\Pr\{\xi = 1\} = 0.4$, $\Pr\{\xi = 2\} = 0.3$, and $\Pr\{\xi = 3\} = 0.2$. Set up the corresponding transition probability matrix for the end-of-period inventory level X_n.

3.3.5 An urn initially contains a single red ball and a single green ball. A ball is drawn at random, removed, and replaced by a ball of the opposite color, and this process repeats so that there are always exactly two balls in the urn. Let X_n be the number of red balls in the urn after n draws, with $X_0 = 1$. Specify the transition probabilities for the Markov chain $\{X_n\}$.

Problems

3.3.1 An urn contains six tags, of which three are red and three are green. Two tags are selected from the urn. If one tag is red and the other is green, then the selected tags are discarded and two blue tags are returned to the urn. Otherwise, the selected tags are resumed to the urn. This process repeats until the urn contains only blue tags. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 3$. (This is an elementary model of a chemical reaction in which red and green atoms combine to form a blue molecule.) Give the transition probability matrix.

3.3.2 Three fair coins are tossed, and we let X_1 denote the number of heads that appear. Those coins that were heads on the first trial (there were X_1 of them) we pick up and toss again, and now we let X_2 be the total number of tails, including those left from the first toss. We toss again all coins showing tails, and let X_3 be the resulting total number of heads, including those left from the previous toss. We continue the process. The pattern is, count heads, toss heads, count tails, toss tails, count heads, toss heads, etc., and $X_0 = 3$. Then, $\{X_n\}$ is a Markov chain. What is the transition probability matrix?

3.3.3 Consider the inventory model of Section 3.3.1. Suppose that unfulfilled demand is not back ordered but is lost.

(a) Set up the corresponding transition probability matrix for the end-of-period inventory level X_n.

(b) Express the long run fraction of lost demand in terms of the demand distribution and limiting probabilities for the end-of-period inventory.

3.3.4 Consider the queueing model of Section 3.4. Now, suppose that at most a single customer arrives during a single period, but that the service time of a customer is a random variable Z with the geometric probability distribution

$$\Pr\{Z = k\} = \alpha(1-\alpha)^{k-1} \quad \text{for } k = 1, 2, \ldots.$$

Specify the transition probabilities for the Markov chain whose state is the number of customers waiting for service or being served at the start of each period. Assume that the probability that a customer arrives in a period is β and that no customer arrives with probability $1-\beta$.

3.3.5 You are going to successively flip a quarter until the pattern HHT appears, that is, until you observe two successive heads followed by a tails. In order to calculate some properties of this game, you set up a Markov chain with the following states: $0, H, HH$, and HHT, where 0 represents the starting point, H represents a single observed head on the last flip, HH represents two successive heads on the last two flips, and HHT is the sequence that you are looking for. Observe that if you have just tossed a tails, followed by a heads, a next toss of a tails effectively starts you over again in your quest for the HHT sequence. Set up the transition probability matrix.

3.3.6 Two teams, A and B, are to play a best of seven series of games. Suppose that the outcomes of successive games are independent, and each is won by A with probability p and won by B with probability $1-p$. Let the state of the system be represented by the pair (a, b), where a is the number of games won by A, and b is the number of games won by B. Specify the transition probability matrix. Note that $a + b \leq 7$ and that the entries end whenever $a = 4$ or $b = 4$.

3.3.7 A component in a system is placed into service, where it operates until its failure, whereupon it is replaced *at the end of the period* with a new component having statistically identical properties, and the process repeats. The probability that a component lasts for k periods is α_k, for $k = 1, 2, \ldots$. Let X_n be the remaining life of the component in service *at the end-of-period* n. Then, $X_n = 0$ means that X_{n+1} will be the total operating life of the next component. Give the transition probabilities for the Markov chain $\{X_n\}$.

3.3.8 Two urns A and B contain a total of N balls. Assume that at time t, there were exactly k balls in A. At time $t+1$, an urn is selected at random in proportion to its contents (i.e., A is chosen with probability k/N and B is chosen with

f heads, including those left from the
he pattern is, count heads, toss heads,
eads, etc., and $X_0 = 3$. Then, $\{X_n\}$ is a
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st demand in terms of the demand dis-
. for the end-of-period inventory.
n 3.4. Now, suppose that at most a single
d, but that the service time of a customer
etric probability distribution

$= 1, 2, \ldots.$

for the Markov chain whose state is the
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that a customer arrives in a period is β and
bility $1 - \beta$.
a quarter until the pattern HHT appears,
ssive heads followed by a tails. In order to
game, you set up a Markov chain with the
IT, where 0 represents the starting point, H
n the last flip, HH represents two successive
HT is the sequence that you are looking for.
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again in your quest for the HHT sequence.
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a best of seven series of games. Suppose that
s are independent, and each is won by A with
probability $1 - p$. Let the state of the system
), where a is the number of games won by A,
won by B. Specify the transition probability
that the entries end whenever $a = 4$ or $b = 4$.
ced into service, where it operates until its fail-
t the end of the period with a new component
roperties, and the process repeats. The proba-
for k periods is α_k, for $k = 1, 2, \ldots$. Let X_n be
ponent in service *at the end-of-period* n. Then,
e the total operating life of the next component.
es for the Markov chain $\{X_n\}$.
otal of N balls. Assume that at time t, there were
$t+1$, an urn is selected at random in proportion
osen with probability k/N and B is chosen with

probability $(N-k)/N$). Then, a ball is selected from A with probability p or from B with probability q and placed in the previously chosen urn. Determine the transition matrix for this Markov chain.

3.3.9 Suppose that two urns A and B contain a total of N balls. Assume that at time t, there are exactly k balls in A. At time $t+1$, a ball and an urn are chosen with probability depending on the contents of the urn (i.e., a ball is chosen from A with probability k/N or from B with probability $(N-k)/N$). Then, the ball is placed into one of the urns, where urn A is chosen with probability k/N or urn B is chosen with probability $(N-k)/N$. Determine the transition matrix of the Markov chain with states represented by the contents of A.

3.3.10 Consider a discrete-time, periodic review inventory model and let ξ_n be the total demand in period n, and let X_n be the inventory quantity on hand at the end-of-period n. An (s, S) inventory policy is used: If the end-of-period stock is not greater than s, then a quantity is instantly procured to bring the level up to S. If the end-of-period stock exceeds s, then no replenishment takes place.

(a) Suppose that $s = 1, S = 4$, and $X_0 = S = 4$. If the period demands turn out to be $\xi_1 = 2, \xi_2 = 3, \xi_3 = 4, \xi_4 = 0, \xi_5 = 2, \xi_6 = 1, \xi_7 = 2$, and $\xi_8 = 2$, what are the end-of-period stock levels X_n for periods $n = 1, 2, \ldots, 8$?

(b) Suppose $\xi_1, \xi_2, \ldots$ are independent random variables where $\Pr\{\xi_n = 0\} = 0.1, \Pr\{\xi_n = 1\} = 0.3, \Pr\{\xi_n = 2\} = 0.3, \Pr\{\xi_n = 3\} = 0.2$, and $\Pr\{\xi_n = 4\} = 0.1$. Then, $X_0, X_1, \ldots$ is a Markov chain. Determine P_{41} and P_{04}.

3.4 First Step Analysis

A surprising number of functionals on a Markov chain can be evaluated by a technique that we call *first step analysis*. This method proceeds by analyzing, or breaking down, the possibilities that can arise at the end of the first transition, and then invoking the law of total probability coupled with the Markov property to establish a characterizing relationship among the unknown variables. We first applied this technique in Theorem 3.1. In this section, we develop a series of applications of the technique.

3.4.1 *Simple First Step Analyses*

Consider the Markov chain $\{X_n\}$ whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{array} \right\| \end{array},$$

where $\alpha > 0, \beta > 0, \gamma > 0$, and $\alpha + \beta + \gamma = 1$. If the Markov chain begins in state 1, it remains there for a random duration and then proceeds either to state 0 or to state 2, where it is trapped or absorbed. That is, once in state 0, the process remains there for

The other states give us

$$\begin{aligned} w_3 &= 0.4w_2 + 0.5w_3 + 0.1w_5, \\ w_4 &= 0.4w_2 + 0.5w_4 + 0.1w_5, \\ w_5 &= w_5. \end{aligned}$$

Since state E_5 corresponds to death, it is clear that we must have $w_5 = 0$. With this prescription, the reduced equations become, after elementary simplification,

$$\begin{array}{rcl} -1.0w_0 + 0.9w_1 & = & 0, \\ -0.5w_1 + 0.4w_2 & = & 0, \\ -0.4w_2 + 0.2w_3 + 0.1w_4 & = & -1, \\ 0.4w_2 - 0.5w_3 & = & 0, \\ 0.4w_2 \qquad - 0.5w_4 & = & 0. \end{array}$$

The unique solution is

$$w_0 = 4.5, \quad w_1 = 5.00, \quad w_2 = 6.25, \quad w_3 = w_4 = 5.00.$$

Each female, on the average, spends $w_0 = W_{02} = 4.5$ periods in the childbearing state E_2 during her lifetime.

Exercises

3.4.1 Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 1 & 0 & 0.7 & 0.2 & 0.1 \\ 2 & 0 & 0 & 0.9 & 0.1 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

3.4.2 Consider the Markov chain whose transition probablity matrix is given by

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0.1 & 0.6 & 0.3 \\ 2 & 0 & 0 & 1 \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

3.4.3 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

3.4.4 A coin is tossed repeatedly until two successive heads appear. Find the mean number of tosses required.

Hint: Let X_n be the cumulative number of successive heads. The state space is $0, 1, 2$, and the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Determine the mean time to reach state 2 starting from state 0 by invoking a first step analysis.

3.4.5 A coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose the first coin toss results in a head. Find the probability that the game ends with two successive tails.

3.4.6 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

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probability that the Markov chain ends in

rption.
o successive heads appear. Find the mean

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e first coin toss results in a head. Find the prob-
two successive tails.
ose transition probability matrix is given by

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0
0.4
0.1
1

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e to absorption.

3.4.7 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

3.4.8 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

3.4.9 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 1, determine the probability that the process is absorbed into state 0. Compare this with the $(1,0)$th entry in the matrix powers $\mathbf{P}^2, \mathbf{P}^4, \mathbf{P}^8$, and $\mathbf{P}^{16}$.

Problems

3.4.1 Which will take fewer flips, on average: successively flipping a quarter until the pattern *HHT* appears, i.e., until you observe two successive heads followed by a tails; or successively flipping a quarter until the pattern *HTH* appears? Can you explain why these are different?

3.4.2 A zero-seeking device operates as follows: If it is in state m at time n, then at time $n+1$, its position is uniformly distributed over the states $0, 1, \ldots, m-1$. Find the expected time until the device first hits zero starting from state m.

Note: This is a highly simplified model for an algorithm that seeks a maximum over a finite set of points.

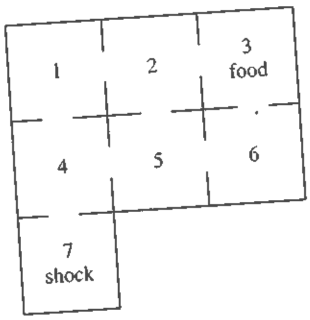
3.4.3 A zero-seeking device operates as follows: If it is in state j at time n, then at time $n+1$, its position is 0 with probability $1/j$, and its position is k (where k is one of the states $1, 2, \ldots, j-1$) with probability $2k/j^2$. Find the expected time until the device first hits zero starting from state m.

3.4.4 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{array} \right\| \end{array}.$$

Starting in state $X_0 = 1$, determine the probability that the process never visits state 2. Justify your answer.

3.4.5 A white rat is put into compartment 4 of the maze shown here:



It moves through the compartments at random; i.e., if there are k ways to leave a compartment, it chooses each of these with probability $1/k$. What is the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7?

3.4.6 Consider the Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array},$$

where $p+q=1$. Determine the mean time to reach state 4 starting from state 0. That is, find $E[T|X_0=0]$, where $T=\min\{n\geq 0; X_n=4\}$.

Hint: Let $v_i=E[T|X_0=i]$ for $i=0,1,\ldots,4$. Establish equations for $v_0,v_1,\ldots,v_4$ by using a first step analysis and the boundary condition $v_4=0$. Then, solve for v_0.

3.4.7 Let X_n be a Markov chain with transition probabilities P_{ij}. We are given a "discount factor" β with $0<\beta<1$ and a cost function $c(i)$, and we wish to determine the total expected discounted cost starting from state i, defined by

$$h_i=E\left[\sum_{n=0}^{\infty}\beta^n c(X_n)|X_0=i\right].$$

Using a first step analysis show that h_i satisfies the system of linear equations

$$h_i=c(i)+\beta\sum_j P_{ij}h_j \quad \text{for all states } i.$$

3.4.8 An urn contains five red and three green balls. The balls are chosen at random, one by one, from the urn. If a red ball is chosen, it is removed. Any green ball that is chosen is returned to the urn. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

3.4.9 An urn contains five red and three yellow balls. The balls are chosen at random, one by one, from the urn. Each ball removed is replaced in the urn by a yellow ball. The selection process continues until all of the red balls have been removed from the urn. What is the mean duration of the game?

3.4.10 You have five fair coins. You toss them all so that they randomly fall heads or tails. Those that fall tails in the first toss you pick up and toss again. You toss again those that show tails after the second toss, and so on, until all show heads. Let X be the number of coins involved in the *last* toss. Find $\Pr\{X=1\}$.

3.4.11 An urn contains two red and two green balls. The balls are chosen at random, one by one, and removed from the urn. The selection process continues until all of the green balls have been removed from the urn. What is the probability that a single red ball is in the urn at the time that the last green ball is chosen?

3.4.12 A Markov chain $X_0,X_1,X_2,\ldots$ has the transition probability matrix

$$\mathbf{P}=\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\begin{array}{c}\begin{array}{ccc}0 & 1 & 2\end{array}\\ \left\|\begin{array}{ccc}0.3 & 0.2 & 0.5\\ 0.5 & 0.1 & 0.4\\ 0 & 0 & 1\end{array}\right\|\end{array}$$

and is known to start in state $X_0=0$. Eventually, the process will end up in state 2. What is the probability that when the process moves into state 2, it does so from state 1?

Hint: Let $T = \min\{n \geq 0; X_n = 2\}$, and let

$$z_i = \Pr\{X_{T-1} = 1 | X_0 = i\} \quad \text{for } i = 0, 1.$$

Establish and solve the first step equations

$$\begin{aligned} z_0 &= \quad 0.3z_0 + 0.2z_1, \\ z_1 &= 0.4 + 0.5z_0 + 0.1z_1. \end{aligned}$$

3.4.13 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that the time $T = \min\{n \geq 0; X_n = 2\}$ is an odd number?

3.4.14 A single die is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 5 or 7. What is the probability that the game stops at a sum of 5?

3.4.15 A simplified model for the spread of a rumor goes this way: There are $N = 5$ people in a group of friends, of which some have heard the rumor and the others have not. During any single period of time, two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friends is just as likely as between any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n denote the number of friends who have heard the rumor at the end of the nth period.

Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

3.4.16 An urn contains five tags, of which three are red and two are green. A tag is randomly selected from the urn and replaced with a tag of the opposite color. This continues until only tags of a single color remain in the urn. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 3$. What is the probability that the game ends with the urn containing only red tags?

3.4.17 The *damage* X_n of a system subjected to wear is a Markov chain with the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

d let

$= 0, 1.$

ions

e transition probability matrix

$= 0$. Eventually, the process will end up in
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with the urn containing only red tags?
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The system starts in state 0 and fails when it first reaches state 2. Let $T = \min\{n \geq 0; X_n = 2\}$ be the time of failure. Use a first step analysis to evaluate $\phi(s) = E\left[s^T\right]$ for a fixed number $0 < s < 1$. (This is called the *generating function* of T. See Section 3.9.)

3.4.18 *Time-dependent transition probabilities.* A well-disciplined man, who smokes exactly one half of a cigar each day, buys a box containing N cigars. He cuts a cigar in half, smokes half, and returns the other half to the box. In general, on a day in which his cigar box contains w whole cigars and h half cigars, he will pick one of the $w + h$ smokes at random, each whole and half cigar being equally likely, and if it is a half cigar, he smokes it. If it is a whole cigar, he cuts it in half, smokes one piece, and returns the other to the box. What is the expected value of T, the day on which the last whole cigar is selected from the box?

Hint: Let X_n be the number of whole cigars in the box after the nth smoke. Then, X_n is a Markov chain whose transition probabilities vary with n. Define $v_n(w) = E[T|X_n = w]$. Use a first step analysis to develop a recursion for $v_n(w)$ and show that the solution is

$$v_n(w) = \frac{2Nw + n + 2w}{w + 1} - \sum_{k=1}^{w} \frac{1}{k},$$

whence

$$E[T] = v_0(N) = 2N - \sum_{k=1}^{N} \frac{1}{k}.$$

3.4.19 *Computer Challenge.* Let N be a positive integer and let $Z_1, \ldots, Z_N$ be independent random variables, each having the geometric distribution

$$\Pr\{Z = k\} = \left(\frac{1}{2}\right)^k, \quad \text{for } k = 1, 2, \ldots.$$

Since these are discrete random variables, the maximum among them may be unique, or there may be ties for the maximum. Let p_N be the probability that the maximum is unique. How does p_N behave when N is large? (Alternative formulation: You toss N dimes. Those that are heads you set aside; those that are tails you toss again. You repeat this until all of the coins are heads. Then, p_N is the probability that the last toss was of a single coin.)

3.5 Some Special Markov Chains

We introduce several particular Markov chains that arise in a variety of applications.

integers with transition probability

(3.44)

for $i = 0, 1, 2, \ldots$. The zero state plays a
n one transition from any other state, while

n applications and at the same time is very
illustrate concepts and results in terms of it.
arises when one is dealing with success runs
ich admits two possible outcomes, success S
quence of trials with two possible outcomes,
al, the probability of S is α and the probability
of length r happened at trial n if the outcomes
he present trial as the last, were respectively
sent state of the process by the length of the
icular, if the last trial resulted in a failure, then
eceding $r+1$ trials in order have the outcomes
carry the label r. The process is clearly Markov
endent of each other), and its transition matrix

x for $n = 0, 1, 2, \ldots$.

the *current age* in a *renewal process*. Consider a
in discrete units, is a random variable ξ, where

$$, 2, \ldots, \sum_{k=1}^{\infty} a_k = 1.$$

v one when it burns out. Suppose the first bulb la
l time $\xi_1 + \xi_2$, and the nth bulb until time $\xi_1 + \cdots$
s $\xi_1, \xi_2, \ldots$ are independent random variables each
. Let X_n be the age of the bulb in service at time
ed in Figure 3.2.

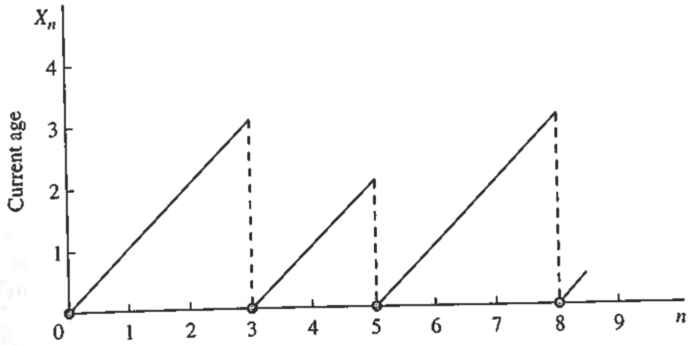

Figure 3.2 The current age X_n in a renewal process. Here, $\xi_1 = 3, \xi_2 = 2$, and $\xi_3 = 3$.

By convention, we set $X_n = 0$ at the time of a failure.

The current age is a success run Markov process for which

$$p_k = \frac{a_{k+1}}{a_{k+1} + a_{k+2} + \cdots}, \quad r_k = 0, q_k = 1 - p_k, \quad \text{for } k = 0, 1, \ldots. \tag{3.45}$$

We reason as follows: The age process reverts to zero upon failure of the item in service. Given that the age of the item in current service is k, then failure occurs in the next time period with *conditional probability* $p_k = a_{k+1}/(a_{k+1} + a_{k+2} + \cdots)$. Given that the item has survived k periods, it survives at least to the next period with the remaining probability $q_k = 1 - p_k$.

Renewal processes are extensively discussed in Chapter 7.

Exercises

3.5.1 The probability of the thrower winning in the dice game called "craps" is $p = 0.4929$. Suppose Player A is the thrower and begins the game with \$5, and Player B, his opponent, begins with \$10. What is the probability that Player A goes bankrupt before Player B? Assume that the bet is \$1 per round.

Hint: Use equation (3.42).

3.5.2 Determine the gambler's ruin probability for Player A when both players begin with \$50, bet \$1 on each play, and where the win probability for Player A in each game is

(a) $p = 0.49292929$

(b) $p = 0.5029237$

(See Chapter 2, Section 2.2.)

What are the gambler's ruin probabilities when each player begins with \$500?

3.5.3 Determine $\mathbf{P}^n$ for $n = 2, 3, 4, 5$ for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{Vmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{Vmatrix}.$$

3.5.4 A coin is tossed repeatedly until three heads in a row appear. Let X_n record the current number of successive heads that have appeared. That is, $X_n = 0$ if the nth toss resulted in tails; $X_n = 1$ if the nth toss was heads and the $(n-1)$st toss was tails; and so on. Model X_n as a success runs Markov chain by specifying the probabilities p_i and q_i.

3.5.5 Suppose that the items produced by a certain process are each graded as defective or good and that whether or not a particular item is defective or good depends on the quality of the previous item. To be specific, suppose that a defective item is followed by another defective item with probability 0.80, whereas a good item is followed by another good item with probability 0.95. Suppose that the initial (zeroth) item is good. Using equation (3.31), determine the probability that the eighth item is good, and verify this by computing the eighth matrix power of the transition probability matrix.

3.5.6 A baseball trading card that you have for sale may be quite valuable. Suppose that the successive bids $\xi_1, \xi_2, \ldots$ that you receive are independent random variables with the geometric distribution

$$\Pr\{\xi = k\} = 0.01(0.99)^k \quad \text{for } k = 0, 1, \ldots.$$

If you decide to accept any bid over \$100, how many bids, on the average, will you receive before an acceptable bid appears?

Hint: Review the discussion surrounding equation (3.35).

3.5.7 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \end{array}.$$

Starting in state 1, determine the probability that the process is absorbed into state 0. Do this first using the basic first step approach of equations (3.21) and (3.22) and second using the particular results for a random walk given in equation (3.42).

3.5.8 As a special case, consider a discrete-time queueing model in which at most a single customer arrives in any period and at most a single customer completes service. Suppose that in any single period, a single customer arrives with probability α, and no customers arrive with probability $1 - \alpha$. Provided that there are customers in the system, in a single period a single customer completes service with probability β, and no customers leave with probability $1-\beta$. Then X_n, the number of customers in the system at the end-of-period n, is a random walk in the sense of Section 3.5.3. Referring to equation (3.38), specify the transition probabilities p_i, q_i, and r_i for $i = 0, 1, \ldots$.

3.5.9 In a simplified model of a certain television game show, suppose that the contestant, having won k dollars, will at the next play have $k+1$ dollars with probability q and be put out of the game and leave with nothing with probability $p = 1 - q$. Suppose that the contestant begins with one dollar. Model her winnings after n plays as a success runs Markov chain by specifying the transition probabilities p_i, q_i, and r_i in equation (3.44).

Problems

3.5.1 As a special case of the successive maxima Markov chain whose transition probabilities are given in equation (3.34), consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 + a_1 & a_2 & a_3 \\ 0 & 0 & a_0 + a_1 + a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}.$$

Starting in state 0, show that the mean time until absorption is $v_0 = 1/a_3$.

3.5.2 A component of a computer has an active life, measured in discrete units, that is a random variable T, where $\Pr\{T = k\} = a_k$ for $k = 1, 2, \ldots$. Suppose one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n. Then, $\{X_n\}$ is a success runs Markov chain.

(a) Specify the probabilities p_i and q_i.

(b) A "planned replacement" policy calls for replacing the component upon its failure or upon its reaching age N, whichever occurs first. Specify the success runs probabilities p_i and q_i under the planned replacement policy.

3.5.3 *A Batch Processing Model.* Customers arrive at a facility and wait there until K customers have accumulated. Upon the arrival of the Kth customer, all are instantaneously served, and the process repeats. Let $\xi_0, \xi_1, \ldots$ denote the arrivals in successive periods, assumed to be independent random variables whose distribution is given by

$$\Pr\{\xi_k = 0\} = \alpha, \quad \Pr\{\xi_k = 1\} = 1 - \alpha,$$

where $0 < \alpha < 1$. Let X_n denote the number of customers in the system at time n. Then, $\{X_n\}$ is a Markov chain on the states $0, 1, \ldots, K-1$. With $K = 3$, give the transition probability matrix for $\{X_n\}$. Be explicit about any assumptions you make.

3.5.4 Martha has a fair die with the usual six sides. She throws the die and records the number. She throws the die again and adds the second number to the first. She repeats this until the cumulative sum of all the tosses first exceeds 10. What is the probability that she stops at a cumulative sum of 13?

3.5.5 Let $\{X_n\}$ be a random walk for which zero is an absorbing state and such that from a positive state, the process is equally likely to go up or down one unit. The transition probability matrix is given by (3.38) with $r_0 = 1$ and $p_i = q_i = \frac{1}{2}$ for $i \geq 1$. (a) Show that $\{X_n\}$ is a nonnegative martingale. (b) Use the maximal inequality in Chapter 2, (2.53) to limit the probability that the process ever gets as high as $N > 0$.

3.6 Functionals of Random Walks and Success Runs

Consider first the random walk on $N+1$ states whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & \cdots & N \\ \hline 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & q & 0 & p & 0 & \cdots & 0 \\ 2 & 0 & q & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ N & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}.$$

"Gambler's ruin" is the event that the process reaches state 0 before reaching state N. This event can be stated more formally if we introduce the concept of *hitting time*. Let T be the (random) time that the process first reaches, or hits, state 0 or N. In symbols,

$$T = \min\{n \geq 0; X_n = 0 \text{ or } X_n = N\}.$$

The random time T is shown in Figure 3.3 in a typical case.

In terms of T, the event written as $X_T = 0$ is the event of gambler's ruin, and the probability of this event starting from the initial state k is

$$u_k = \Pr\{X_T = 0 | X_0 = k\}.$$

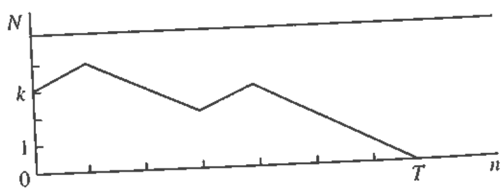

Figure 3.3 The hitting time to 0 or N. As depicted here, state 0 was reached first.