and refer to $\mathbf{P} = \|P_{ij}\|$ as the Markov matrix or *transition probability matrix* of the process.

The ith row of $\mathbf{P}$, for $i = 0, 1, \ldots$, is the probability distribution of the values of X_{n+1} under the condition that $X_n = i$. If the number of states is finite, then $\mathbf{P}$ is a finite square matrix whose order (the number of rows) is equal to the number of states. Clearly, the quantities P_{ij} satisfy the conditions

$$P_{ij} \geq 0 \quad \text{for } i, j = 0, 1, 2, \ldots, \tag{3.3}$$

$$\sum_{j=0}^{\infty} P_{ij} = 1 \quad \text{for } i = 0, 1, 2, \ldots. \tag{3.4}$$

The condition (3.4) merely expresses the fact that some transition occurs at each trial. (For convenience, one says that a transition has occurred even if the state remains unchanged.)

A Markov process is completely defined once its transition probability matrix and initial state X_0 (or, more generally, the probability distribution of X_0) are specified. We shall now prove this fact.

Let $\Pr\{X_0 = i\} = p_i$. It is enough to show how to compute the quantities

$$\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}, \tag{3.5}$$

since any probability involving $X_{j_1}, \ldots, X_{j_k}$, for $j_1 < \cdots < j_k$, can be obtained, according to the axiom of total probability, by summing terms of the form (3.5).

By the definition of conditional probabilities, we obtain

$$\begin{aligned} &\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} \\ &\quad = \Pr\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \\ &\qquad \times \Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\}. \end{aligned} \tag{3.6}$$

Now, by the definition of a Markov process,

$$\begin{aligned} &\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} \\ &\quad = \Pr\{X_n = i_n | X_{n-1} = i_{n-1}\} = P_{i_{n-1}, i_n}. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) gives

$$\begin{aligned} &\Pr\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} \\ &\quad = \Pr\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\} P_{i_{n-1}, i_n}. \end{aligned}$$

Then, upon repeating the argument $n - 1$ additional times, (3.5) becomes

$$\begin{aligned} &\Pr\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} \\ &\quad = p_{i_0} P_{i_0, i_1} \cdots P_{i_{n-2}, i_{n-1}} P_{i_{n-1}, i_n}. \end{aligned} \tag{3.8}$$

This shows that all finite-dimensional probabilities are specified once the transition probabilities and initial distribution are given, and in this sense, the process is defined by these quantities.

Related computations show that (3.1) is equivalent to the Markov property in the form

$$\begin{aligned} &\Pr\{X_{n+1} = j_1, \ldots, X_{n+m} = j_m | X_0 = i_0, \ldots, X_n = i_n\} \\ &\quad = \Pr\{X_{n+1} = j_1, \ldots, X_{n+m} = j_m | X_n = i_n\} \end{aligned} \tag{3.9}$$

for all time points n, m and all states $i_0, \ldots, i_n, j_1, \ldots, j_m$. In other words, once (3.9) is established for the value $m = 1$, it holds for all $m \geq 1$ as well.

Exercises

3.1.1 A Markov chain $X_0, X_1, \ldots$ on states 0, 1, 2 has the transition probability matrix

$$\begin{array}{c} \\ \mathbf{P} = \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = \Pr\{X_0 = 0\} = 0.3, p_1 = \Pr\{X_0 = 1\} = 0.4$, and $p_2 = \Pr\{X_0 = 2\} = 0.3$. Determine $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$.

3.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\begin{array}{c} \\ \mathbf{P} = \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_2 = 1, X_3 = 1 | X_1 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1 | X_0 = 0\}.$$

3.1.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\begin{array}{c} \\ \mathbf{P} = \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$.