02407 Stochastic processes (A2013)

Exercise - A Brownian bridge

Define the Brownian bridge $X_t := W_t - tW_1$, $t \in [0,1]$, where W is a (standard) Brownian motion on $[0,1]$.

i) Prove that $\mathrm{Cov}(W_s, W_t) = s \wedge t$ $(:= \min(s,t))$.
HINT : the covariance operator is bilinear.

ii) Show that if $Y \sim \mathcal{N}_d$, then any transformation of the form $Y \mapsto b + BY$, $b \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times d}$ is $\mathcal{N}_m$.
HINT : $X \sim \mathcal{N}_m \Leftrightarrow \langle a, X \rangle \sim \mathcal{N}_1$ for any $a \in \mathbb{R}^m$

iii) Prove that X is Gaussian.

iv) Prove that $E(X_t) \equiv 0$ and that $\mathrm{Cov}(X_s, X_t) = s \wedge t - st$. Where does $t \mapsto \mathbb{V}ar(X_t)$ attain its maximum? Sketch a typical sample path of X.

v) Prove that X does *not* have independent increments.
HINT : Let $J := \{t_1, ..., t_n\}$, $0 \leq t_1 < ... < t_n \leq 1$. If $t_j \leq t_{i-1}$, what is $\mathrm{Cov}(X_{t_i} - X_{t_{i-1}}, X_{t_j} - X_{t_{j-1}})$?

In continuous time, a process X is said to be a martingale if $E(X_t | \{X_u : u \leq s\}) = X_s$ for $s < t$. If we define $\mathcal{F}_s := \{W_u : u \leq s\}$, the independent increment property of Brownian motion is equivalent to $W_t - W_s$ being independent of the *past*, i.e of $\mathcal{F}_s$.

vi) Prove that W is a martingale, i.e. that $E(W_t | \mathcal{F}_s) = W_s$ for $s < t$.

vii) Prove that X is *not* a martingale.