Lecture 9: Exercises

October 2023

In the following exercises, we investigate a system consisting of 4 identical components, which can either be in functioning or failed state. The lifetimes of the respective components are i.i.d. with distribution:

$$X_i \sim exp(\beta)$$

for $1 \leq i \leq 4$ and $\beta \in \mathbb{R}^+$. Failure is irreversible. The system is inspected at regular intervals of τ time units. The state of the system is given as the number of functioning components at the time of inspection. Thus, the state space is given by:

$$S = \{0, ..., 4\}$$

Without maintenance, the system will deteriorate according to a DTMC.

Exercise 1: Show that the transition probabilities between two inspections are given by:

$$p(t|s) = \binom{s}{t} e^{-\beta\tau t}(1 - e^{-\beta\tau})^{s-t} \tag{1}$$

for $0 \leq t \leq s \leq 4$. *Hint:* Find expressions for $P(X_i \leq \tau) = F(\tau)$ and $P(X_i > \tau) = R(\tau)$. ■

At each inspection, we have the opportunity to replace any number of failed components. Thus, in state $s \in S$, the action set is given by:

$$A_s = \{a_0, ..., a_{4-s}\}$$

where a_i is the action of replacing i components. Failed components can be replaced immediately and are substituted by statistically equivalent (functioning) components.

Exercise 2: Find a general expression for the transition probabilities

$$p(t|s, a_i)$$

for $0 \leq t, s \leq 4$ and $a_i \in A_s$. *Hint:* Use the expression in (1). ■

Assume now, that the cost function is given by:

$$c(s, a_i, t) = \mathbb{1}_{i>0} \cdot \alpha + i \cdot \beta + \mathbb{1}_{t=0} \cdot \gamma$$

for $\alpha, \beta, \gamma \in \mathbb{R}^+$ where $\mathbb{1}_{(\cdot)}$ is the indicator function and t is the realisation of the random variable T representing the successor state (at the next inspection). Here, α and β may represent the fixed and variable cost of replacing components, while γ may be the breakdown/downtime cost.

Exercise 3: Show that the expected cost when taking action a_0 in state $s \in S$ is given by:

$$\bar{c}(s, a_0) := \mathbb{E}\{c(s, a_0, T)\} = \gamma(1 - e^{-\beta\tau})^s$$

What does this say about the cost of systematically leaving the system unmaintained in the long run? ■

We are now given a (stationary deterministic) policy $\pi : S \to A$. Consider the sequence of state-action pairs $S_0, \pi(S_0), S_1, \pi(S_1), S_2, \pi(S_2), ...$ and let

$$c(S_0, \pi(S_0), S_1),\ \ c(S_1, \pi(S_1), S_2),\ \ c(S_2, \pi(S_2), S_3), ...$$

be the sequential costs incurred in the first, second, third etc. decision epochs. To reduce the weight of future costs, we introduce the discount factor $0 < \lambda < 1$ and consider the discounted costs

$$c(S_0, \pi(S_0), S_1),\ \ \lambda c(S_1, \pi(S_1), S_2),\ \ \lambda^2 c(S_2, \pi(S_2), S_3), ...$$

as an alternative cost measure when evaluating long-term plans.

Exercise 4: Show that the *total discounted cost*:

$$\sum_{i=0}^{\infty} \lambda^i c(S_i, \pi(S_i), S_{i+1})$$

over an *infinite horizon* is well-defined. ■

It is now given that the system, when unmaintained, deteriorates according to the DTMC:

$$M_{\emptyset} = \begin{array}{c} \begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \end{array} \\ \left[\begin{array}{ccccc} .0625 & .2500 & .3750 & .2500 & .0625 \\ \cdot & .1250 & .3750 & .3750 & .1250 \\ \cdot & \cdot & .2500 & .5000 & .2500 \\ \cdot & \cdot & \cdot & .5000 & .5000 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \end{array}$$

Assume, that we adopt the (stationary deterministic) maintenance policy $\pi : S \to A$ given by:

$$\pi(4) = \pi(3) = a_0,\ \pi(2) = a_2,\ \pi(1) = a_3,\ \pi(0) = a_4$$

That is, if there are 2 or fewer functioning components, π prescribes replacing all failed components. Else, π prescribes doing nothing (a_0). Implementing a policy reduces an MDP to a DTMC (why?).

Exercise 5: Write up the Markov chain:

$$M_\pi \in \mathbb{R}^{5\times5}$$

induced by the policy π. *Hint:* all the rows in M_π can be read off $M_\emptyset$. ■

Assume now, that the expected cost function $\bar{c}$ takes the following values:

Table 1: Expected cost of state action pairs under π

s	4	3	2	1	0
$\pi(s)$	0	0	2	3	4
$\bar{c}(s, \pi(s))$	625	1,250	2,825	3,425	4,025

Exercise 6: Determine the long run expected cost per epoch under the maintenance policy π. *Hint:* Under the policy π the MDP is reduced to a DTMC which has a limiting distribution. ■