IMM - DTU

02405 Probability
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# Solution for review exercise 20 (chapter 5) in Pitman

**Question a)** This is example 3 page 317. A rederivation gives us

$$P(T_{\min} \leq t) = 1 - P(T_{\min} > t) = 1 - P(T_1 > t, T_2 > t)$$

with $T_1$ and $T_2$ independent we get

$$P(T_{\min} \leq t) = 1 - P(T_1 > t)P(T_2 > t)$$

now inserting the exponential survival function page 279 we get

$$P(T_{\min} \leq t) = 1 - \left(1 - \left(1 - e^{-\lambda_1 t}\right)\right)\left(1 - \left(1 - e^{-\lambda_2 t}\right)\right) = 1 - e^{-(\lambda_1+\lambda_2)t}$$

the cumulative distribution function of an exponentially distributed random variable with parameter $\lambda_1 + \lambda_2$.

**Question b)** This question is Example 2 page 352. A slightly different handling of the integrals gives us

$$P(T_1 < T_2) = \int_0^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \mathrm{d}t_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \int_0^\infty f_{T_1}(t_1) P(T_2 > t_1) \mathrm{d}t_1$$

which is an application of the rule of averaged conditional probability (page 41) for a continuous density. The general result is stated page 417 as the Integral Conditioning Formula. We get

$$P(T_1 < T_2) = \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

**Question c)** Consider

$$P(T_{\min} > t | X_{\min} = 2) = P(T_1 > t | T_2 > T_1) = \frac{P(T_1 > t, T_2 > T_1)}{P(T_2 > T_1)} = \frac{P(T_1 > t, T_2 > T_1)}{P(X_{\min} = 2)}$$

We evaluate the probability in the denominator by integrating the joint density over a proper region (page 349), similarly to example 2 page 352

$$P(T_1 > t, T_2 > T_1) = \int_t^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \mathrm{d}t_1$$

$$= \int_t^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

By inserting back we finally get

$$P(T_{\text{min}} > t | X_{\text{min}} = 2) = e^{-(\lambda_1+\lambda_2)t} = P(T_{\text{min}} > t)$$

such that $T_{\text{min}}$ and $X_{\text{min}}$ are independent.

**Question d)** We can define $X_{\text{min}} = i$ whenever $T_{\text{min}} = T_i$. Then $P(X_{\text{min}} = i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$, and $T_{\text{min}}$ and $X_{\text{min}}$ are independent.