IMM - DTU

02405 Probability
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BFN/bfn

Solution for review exercise 34 (chapter 3) in Pitman

Question a) The function $g_z(x) = z^x$ defines a function of x for any $|z| < 1$. For fixed z we can find the $E(g_z(X))$ using the definition in the box on the top of page 175. We find

$$E(g_z(X)) = E(z^X) = \sum_{x=0}^{\infty} z^x P(X = x)$$

However, this is a power series in z that is absolutely convergent for $|z| \leq 1$ and thus defines a C^∞ function of z for $|z| < 1$.

Question b) The more elegant and maybe more abstract proof is

$$G_{X+Y}(z) = E\left(z^{X+Y}\right) = E\left(z^X z^Y\right)$$

From the independence of X and Y we get (page 177)

$$G_{X+Y}(z) == E\left(z^X\right) E\left(z^Y\right) = G_X(z) G_Y(z)$$

The more crude analytic proof goes as follows

$$G_{X+Y}(z) = E\left(z^{X+Y}\right) = \sum_{k=0}^{\infty} z^k P(X+Y = k) = \sum_{k=0}^{\infty} z^k \left(\sum_{i=0}^{k} P(X = i, Y = k - i) \right)$$

again from the independence of X and Y we get

$$G_{X+Y}(z) = \sum_{k=0}^{\infty} z^k \left(\sum_{i=0}^{k} P(X = i) P(Y = k - i) \right) \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} z^k P(X = i) P(Y = k-i)$$

The interchange of the sums are justified since all terms are positive. The rearrangement is a commonly used tool in analytic derivations in probability. It is quite instructive to draw a small diagram to verify the limits of the sums. We now make further rearrangements

$$G_{X+Y}(z) = \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} z^k P(X = i) P(Y = k - i)$$

$$= \sum_{i=0}^{\infty} z^i P(X = i) \sum_{k=i}^{\infty} z^{k-i} P(Y = k - i) = \sum_{i=0}^{\infty} z^i P(X = i) \sum_{m=0}^{\infty} z^m P(Y = m)$$

by a change of variable ($m = k - i$). Now

$$G_{X+Y}(z) = \sum_{i=0}^{\infty} z^i P(X = i) \sum_{m=0}^{\infty} z^m P(Y = m) = \sum_{i=0}^{\infty} z^i P(X = i) G_Y(z) = G_X(z) G_Y(z)$$

Question c) By rearranging $S_n = (X_1 + \cdots + X_{n-1}) + X_n$ we deduce

$$G_{S_n}(z) = \prod_{i=1}^{n} G_{X_i}(z)$$

We first find the generating function of a Bernoulli distributed random variable(binomial with $n = 1$)

$$E(z^X) = \sum_{x=0}^{1} z^x P(X = x) = z^0 \cdot (1 - p) + z^1 \cdot p = 1 - p(1 - z)$$

Now using the general result for X_i with binomial distribution $b(n_i, p)$ we get

$$E(z^{X_i}) = (E(z^X))^{n_i} = (1 - p(1 - z))^{n_i}$$

Generalizing this result we find

$$E(z^{S_n}) = (1 - p(1 - z))^{\sum_{i=1}^{n} n_i}$$

i.e. that the sum of independent binomially distributed random variables is itself binomially distributed provided equality of the p_i's.

Question d) The generating function of the Poisson distribution is given in exercise 3.5.19. Such that

$$G_{S_n}(z) = \prod_{i=1}^{n} e^{-\mu_i(1-z)} = e^{-\sum_{i=1}^{n} \mu_i(1-z)}$$

The result proofs that the sum of independent Poisson random variables is itself Poisson.

Question e)

$$G_X(z) = \frac{zp}{1 - z(1 - p)} \qquad G_{S_n} = \left(\frac{zp}{1 - z(1 - p)}\right)^n$$

Question f)

$$G_{S_n} = \left(\frac{zp}{1 - z(1 - p)}\right)^{\sum_{i=1}^{n} r_i}$$