IMM - DTU

02405 Probability
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BFN/bfn

Solution for review exercise 29 (chapter 3) in Pitman

Question a) We note that the probability does not depend on the ordering, i.e. the probability of a certain sequence depends on the number of 1's among the X_i's not on the ordering.

$$\frac{\prod_{j=0}^{k-1}(b+jd)\prod_{j=0}^{n-k-1}(w+jd)}{\prod_{j=0}^{k-1}(b+w+jd)}$$

Question b) To obtain the distribution of S_n the number of black balls drawn, we note that there is $\left(\begin{array}{c} n \\ k \end{array}\right)$ different sequences each with the probability derived in question a) that lead to the event $S_n = k$.

$$P(S_n = k) = \left(\begin{array}{c} n \\ k \end{array}\right) \frac{\prod_{j=0}^{k-1}(b+jd)\prod_{j=0}^{n-k-1}(w+jd)}{\prod_{j=0}^{k-1}(b+w+jd)}$$

Question c)

$$\left(\begin{array}{c} n \\ k \end{array}\right) \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}$$

Question d) Not independent since, but interchangeable

Question e) We approach the question by induction. We first show

$$P(X_1 = 1) = \frac{b}{b+w}$$

We then derive $P(X_{n+1} = 1)$ assuming $P(X_n = 1) = bb + w$ in a Polya model.

$$P(X_{n+1} = 1) = P(X_{n+1} = 1|X_1 = 1)P(X_1 = 1) + P(X_{n+1} = 1|X_1 = 0)P(X_1 = 0) = P(X_{n+1} = 1|X_1 =$$

To proceed we note that the probability $P(X_{n+1} = 1|X_1 = 1)$ is the probability of $P(Y_n = 1)$ in an urn scheme starting with $b + d$ blacks and w whites, thus $P(X_{n+1} = 1|X_1 = 1) = P(Y_n = 1) = \frac{b+d}{b+w+d}$. Correspondingly $P(X_{n+1} = 1|X_1 = 0) = \frac{b}{b+w+d}$. Finally

$$P(X_{n+1} = 1) = \frac{b+d}{b+w+d}\frac{b}{b+w} + \frac{b}{b+w+d}\frac{w}{b+w} = \frac{b}{b+w}$$

Question f)

$$P(X_5 = 1|X_{10} = 1) = \frac{P(X_{10} = 1|X_5 = 1)P(X_5 = 1)}{P(X_{10} = 1)} = P(X_{10} = 1|X_5 = 1)$$

using Bayes rule, or from the exchangeability. From the exchangeability we also have

$$P(X_{10} = 1|X_5 = 1) = P(X_2 = 1|X_1 = 1) = \frac{b + d}{b + w + d}$$