IMM - DTU

02405 Probability
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# Solution for review exercise 24 (chapter 3) in Pitman

**Question a)** Following the hint, we write down the permutations of $\{1,2,3\}$

| $X=x$ | $Y=y$ | $Z=z$ | $I(X>Y)$ | $I(Y>Z)$ | $I(Z>X)$ |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 0 | 0 | 1 |
| 1 | 3 | 2 | 0 | 1 | 1 |
| 2 | 1 | 3 | 1 | 0 | 1 |
| 2 | 3 | 1 | 0 | 1 | 0 |
| 3 | 1 | 2 | 1 | 0 | 0 |
| 3 | 2 | 1 | 1 | 1 | 0 |

By picking the three sequences $\{1,3,2\},\{2,1,3\},\{3,2,1\}$ and assigning equal probability $\left(\frac{1}{3}\right)$ to each of them we get

$$P(X>Y) = P((X,Y,Z)\in\{\{2,1,3\},\{3,2,1\}\} = \frac{2}{3}, P(Y>Z) = \frac{2}{3}, P(Z>X) = \frac{2}{3}$$

as we wanted to show.

**Question b)**

$$P(X>Y)+P(Y>Z)+P(Z>X) = E(I_{X>Y})+E(I_{Y>Z})+E(I_{Z>X}) = E(I_{X>Y}+I_{Y>Z}+I_{Z>X})$$

The sum of $I_{X>Y}+I_{Y>Z}+I_{Z>Z}$ can not be greater than 2, thus the smallest of the three probabilities $P(X>Y), P(Y>Z), P(Z>X)$ can not exceed $\frac{2}{3}$.

**Question c)** By a proper mixture of the preferences $A$ for $B$, $B$ for $C$, and $C$ for $A$. Assume that the people in the survey are equally divided among the three possible rankings.

**Question d)** We assign equal probability $\left(\frac{1}{n}\right)$ to the permuations

$$\{n,n-1,\ldots,2,1\},\{1,n,n-1,\ldots,3,2\},\ldots,\{n-1,n-2,\ldots,1,n\}$$

In the sequences $X_1,X_2,\ldots,X_n$, only one of the relations $X_i > X_{i+1}$ will be violoated. (for $i=n$ the relation is $X_n > X_1$).

**Question e)**

$$P(X>Y) = p_1 + (1-p_1)(1-p_2), P(Y>Z) = p_2, P(Z>X) = 1-p_1$$

We can achieve $p = P(X>Y) = P(Y>Z) = P(Z>X)$ for $p_1 = \frac{3-\sqrt{5}}{2}$ and $p_2 = \frac{\sqrt{5}-1}{2}$.