IMM - DTU

02405 Probability
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Solution for review exercise 2.28 (chapter 2) in Pitman

Question a) We define the events A_i that person i receives a correct letter. Each person has a probability of $\frac{1}{n}$ of receiving the correct letter. Thus we have $P(A_i) = \frac{1}{n}$. From the multiplication rule (boxed result at the top of page 37) we have $P(A_i \cap A_j) = P(A_i)P(A_j|A_i)$. Knowing that at person got the right letter, we can conceptually remove this letter from the considerations and rethink the problem with $n-1$ letters. Thus the conditional probability $P(A_j|A_i)$ is $\frac{1}{n-1}$. Generally we can write $P(\cap_{i=1}^k A_i) = P(A_1)P(A_2|A_1)\cdots P(A_k|A_1 \cap A_2 \cdots \cap A_{k-1}) = \frac{1}{n\cdot(n-1)\ldots(n-k+1)} = \frac{(n-k)!}{n!}$. The event that at least one letter is correctly addressed is the union of all the events A_i. From exclusion-inclusion we get

$$P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots (-1)^{n-1} P\left(\cap_{i=1}^n A_i\right)$$

$$= \sum_{i=1}^n \binom{n}{i} (-1)^{i-1} \frac{(n-i)!}{n!} = \sum_{i=1}^n (-1)^{i-1} \frac{1}{i!}$$

Question b) The sum in question a) is close to the first n terms of the Taylor expansion of e^{-1}. Thus approximately for large n

$$P(\cup_{i=1}^n A_i) \tilde{=} 1 - \frac{1}{e}$$