IMM - DTU

02405 Probability
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BFN/bfn

Solution for review exercise 10 (chapter 1) in Pitman

We define the events

E_k Exactly k blood types are represented

Ai i persons have blood type A

Bi i persons have blood type B

Ci i persons have blood type C

Di i persons have blood type D

Question a)

$$P(E_2) = P(A_2) + P(B_2) + P(C_2) + P(D_2) = p_a^2 + p_b^2 + p_c^2 + p_d^2 = 0.3816$$

Question b) We have $p(k) = P(E_k)$. By combinatorial considerations we can show

$$P(A_{i_1} \cap B_{i_2} \cap C_{i_3} \cap D_{i_4}) = \frac{(i_1 + i_2 + i_3 + i_4)!}{i_1! i_2! i_3! i_4!} p_a^{i_1} p_b^{i_2} p_c^{i_3} p_d^{i_4}$$

with $i_1 + i_2 + i_3 + i_4 = 4$, in our case. We have to sum over the appropriate values of (i_1, i_2, i_3, i_4).

It is doable but much more cumbersome to use basic rules. We get

$$p(1) = 0.0687 \qquad p(2) = 0.5973 \qquad p(3) = 0.3163 \qquad p(4) = 0.0177$$

$$p(1) = P(E_1) = P(A_4) + P(B_4) + P(C_4) + P(D_4) = p_a^4 + p_b^4 + p_c^4 + p_d^4 = 0.0687$$
$$p(4) = P(E_4) = P(A_1 \cap B_1 \cap C_1 \cap D_1) = 24 p_a p_b p_c p_d = 0.0177$$

To calculate $p(3) = P(E_3)$ we use the law of averaged conditional probabilities

$$p(3) = P(E_3) = \sum_{i=0}^{4} P(E_3|A_i) P(A_i).$$

We immediately have
$$P(E_3|A_4) = P(E_3|A_3) = 0$$

To establish $P(E_3|A_2)$ we argue

$$P(E_3|A_2) = P(B_1 \cap C_1|A_2) + P(B_1 \cap D_1|A_2) + P(C_1 \cap D_1|A_2) = \frac{p_b p_c + p_b p_d + p_c p_d}{(1-p_a)^2}$$

further

$$P(E_3|A_0) = P(B_2 \cap C_1 \cap D_1|A_0) + P(B_1 \cap C_2 \cap D_1|A_0) + P(B_1 \cap C_1 \cap D_2|A_0) = \frac{4 p_b p_c p_d (p_b + p_c + p_d}{(1-p_a)^4}$$

To evaluate $P(E3|A_1)$ we use the law of averaged conditional probability once more (see Review Exercise 1.13)

$$P(E3|A1) = \sum_{i=1}^{4} P(E3|A_1 \cap B_i) P(B_i|A_1)$$

with

$$P(E3|A_1 \cap B_0) = \frac{3 p_c p_d (p_c + p_d)}{(1 - p_a - p_b)^3}$$

$$P(E3|A_1 \cap B_1) = \frac{p_c^2 + p_d^2}{(1 - p_a - p_b)^2}$$

$$P(E3|A_1 \cap B_2) = \frac{p_c + p_d}{1 - p_a - p_b}$$

$$P(E3|A_1 \cap B_3) = 0$$

and we get

$$P(E3|A1) = \frac{3 p_c p_d (p_c + p_d)}{(1 - p_a - p_b)^3} \left(\frac{1 - p_a - p_b}{1 - p_a} \right)^3 + \frac{p_c^2 + p_d^2}{(1 - p_a - p_b)^2}$$