Udvalgte løsninger til

Probability

(Jim Pitman)

http://www2.imm.dtu.dk/courses/02405/

17. december 2006

IMM - DTU

02405 Probability
2004-2-2
BFN/bfn

Solution for exercise 1.1.1 in Pitman

Question a) $\frac{2}{3}$

Question b) 67%.

Question c) 0.667

Question a.2) $\frac{4}{7}$

Question b.2) 57%.

Question c.2) 0.571

IMM - DTU

02405 Probability
2004-2-2
BFN/bfn

Solution for exercise 1.1.2 in Pitman

Question a) 8 of 11 words has four or more letters: $\frac{8}{11}$

Question b) 4 words have two or more vowels: $\frac{4}{11}$

Question c) The same words qualify (4): $\frac{4}{11}$

Solution for exercise 1.1.7 in Pitman

A special case of a problem, which we will treat in full generality later.

Question a) count the possibilitites 4 out of 36, $\frac{1}{9}$

Question b) count the possibilitites 9 out of 36, $\frac{1}{4}$

Question c) From a) and b) $\frac{1}{4} - \frac{1}{9} = \frac{5}{36}$

Question d) b) in general $\frac{x^2}{36}$ c) in general $\frac{2x-1}{36}$

Question e) The sum is over all possible outcomes, and should thus be 1. Inserting $x = 6$ we get $\frac{6^2}{36} = 1$ q.e.d.

IMM - DTU

02405 Probability
2003-9-10
BFN/bfn

Solution for exercise 1.2.4 in Pitman

It may be useful to read the definition of Odds and *payoff odds* in Pitman pp. 6 in order to solve this exercise

Question a) We define the profit pr

$$pr = 10(8+1) - 100 \cdot 1 = -10$$

Question b) The average gain pr. game is defined as the profit divided by the number of games

$$\frac{pr}{n} = \frac{-10}{100} = -0.1$$

IMM - DTU

02405 Probability
2004-2-4
BFN/bfn

Solution for exercise 1.3.1 in Pitman

Denote the fraction the neighbor gets by x. Then your friend gets $2x$ and you get $4x$. The total is one, thus $x = \frac{1}{7}$ and you get $\frac{4}{7}$.

IMM - DTU

02405 Probability
2004-9-3
BFN/bfn

Solution for exercise 1.3.2 in Pitman

Question a) The event which occurs if exactly one of the events A and B occurs

$$(A \cap B^c) \cup (A^c \cap B)$$

Question b) The event which occurs if none of the events A, B, or C occurs.

$$(A^c \cap B^c \cap C^c)$$

Question c) The events obtained by replacing "none" in the previous question by "exactly one", "exactly two", and "three"

Exactly one $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

Exactly two $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

Exactly three $(A \cap B \cap C)$

IMM - DTU

02405 Probability
2004-2-4
KKA,BFN/bfn,kka

Solution for exercise 1.3.4 in Pitman

We define the outcome space $\Omega = \{0, 1, 2\}$

Question a) yes, $\{0, 1\}$

Question b) yes, $\{1\}$

Question c) no, (we have no information on the sequence)

Question d) yes, $\{1, 2\}$

IMM - DTU

02405 Probability
2003-9-13
KKA/bfn,kka

Solution for exercise 1.3.8 in Pitman

It may be useful to make a sketch similar to the one given at page 22 in Pitman.

From the text the following probabilities are given:

$$P(A) = 0.6 \quad P(A^c) = 1 - P(A) = 0.4$$

$$P(B) = 0.4 \quad P(B^c) = 1 - P(B) = 0.6$$

$$P(AB) = P(A \cap B) = 0.2$$

Question a)

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.6 + 0.4 - 0.2 = 0.8$$

Question b)

$$P(A^c) = 1 - P(A) = 1 - 0.6 = 0.4$$

Question c)

$$P(B^c) = 1 - P(B) = 1 - 0.4 = 0.6$$

Question d)

$$P(A^cB) = P(B) - P(AB) = 0.4 - 0.2 = 0.2$$

Question e)

$$P(A \cup B^c) = 1 - P(B) + P(AB) = 1 - 0.4 + 0.2 = 0.8$$

Question f)

$$P(A^cB^c) = 1 - P(A) - P(B) + P(AB) = 1 - 0.6 - 0.4 + 0.2 = 0.2$$

IMM - DTU

02405 Probability
2003-9-18
BFN/bfn

Solution for exercise 1.3.9 in Pitman

Question a)

$$P(F \cup G) = P(F) + P(G) - P(F \cap G) = 0.7 + 0.6 - 0.4 = 0.9$$

using exclusion-inclusion.

Question b)

$$P(F \cup G \cup H) = P(F) + P(G) + P(H) - P(F \cap G) - P(F \cap H) - P(G \cap H) + P(F \cap G \cap H)$$

$$= 0.7 + 0.6 + 0.5 - 0.4 - 0.3 - 0.2 + 0.1 = 1.0$$

using the general version of exclusion-inclusion (see exercise 1.3.11 and 1.3.12).

Question c)

$$P(F^c \cap G^c \cap H) = P((F \cup G)^c \cap H)$$

$$P(H) = P((F \cup G)^c \cap H) + P((F \cup G) \cap H)$$

The latter probability is

$$P((F \cup G) \cap H) = P((F \cap H) \cup (G \cap H)) = P(F \cap H) + P(G \cap H) - P(F \cap G \cap H)$$

$$= 0.3 + 0.2 - 0.1 = 0.4$$

such that

$$P(F^c \cap G^c \cap H) = 0.5 - 0.4 = 0.1$$

IMM - DTU

02405 Probability
2003-9-11
BFN/bfn

Solution for exercise 1.3.11 in Pitman

$$P(A \cup B \cup C) = P(A \cup (B \cup C))$$

now applying inclusion-exclusion

$$P(A{\cup}(B{\cup}C)) = P(A){+}P(B{\cup}C){-}P(A{\cap}(B{\cup}C)) = P(A){+}P(B{\cup}C){-}P((A{\cap}B){\cup}(A{\cap}C))$$

once again we aplly inclusion-exclusion (the second and the third time) to get

$$P(A{\cup}(B{\cup}C)) = P(A){+}P(B){+}P(C){-}P(B{\cap}C){-}(P(A{\cap}B){+}P(A{\cap}C){-}P((A{\cap}B){\cap}(A{\cap}C)))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

IMM - DTU

02405 Probability
2003-9-11
BFN/bfn

Solution for exercise 1.3.12 in Pitman

We know from exercise 1.3.11 that the formula is valid for $n = 3$ and consider

$$P\left(\cup_{i=1}^{n+1} A_i\right) = P\left(\left(\cup_{i=1}^{n} A_i\right) \cup A_{n+1}\right).$$

Using exclusion-inclusion for two events we get the formula stated p.32. Since the exclusion-inclusion formula is assumed valid for n events we can use this formula for the first term. To get through we realize that the last term

$$P\left(\cup_{i=1}^{n} A_i A_{n+1}\right)$$

is of the form

$$P\left(\cup_{i=1}^{n} B_i\right)$$

with $B_i = A_i \cap A_{n+1}$, implying that we can use the inclusion-exclusion formula for this term too. The proof is completed by writing down the expansion explicitly.

IMM - DTU

02405 Probability
2004-2-4
BFN/bfn

Solution for exercise 1.4.1 in Pitman

Question a) Can't be decided we need to know the proportions of women and men (related to the averaging of conditional probabilities p. 41)

Question b) True, deduced from the rule of averaged conditional probabilities

Question c) True

Question d) True

Question e)

$$\frac{3}{4} \cdot 0.92 + \frac{1}{4} \cdot 0.88 = 0.91$$

true

IMM - DTU

02405 Probability
2003-9-13
KKA/bfn,kka

Solution for exercise 1.4.2 in Pitman

We define the events

A The light bulb is not defect

B The light bulb is produced in city B

From the text the following probabilities are given:

$$P(A|B) = 0.99 \quad P(A^c|B) = 1 - P(A|B) = 0.01$$

$$P(B) = 1/3 \quad P(B^c) = 2/3$$

solution

$$P(A \cap B) = P(B)P(A|B) = 0.99/3 = 0.33$$

IMM - DTU

02405 Probability
2003-9-24
BFN/bfn

Solution for exercise 1.4.9 in Pitman

Question a) In scheme A all 1000 students have the same probability ($\frac{1}{1000}$) of being chosen. In scheme B the probability of being chosen depends on the school. A student from the first school will be chosen with probability $\frac{1}{300}$, from the second with probability $\frac{1}{1200}$, and from the third with probability $\frac{1}{1500}$. The probability of chosing a student from school 1 is $p_1 \cdot \frac{1}{100}$, thus $p_1 = \frac{1}{10}$. Similarly we find $p_2 = \frac{2}{5}$ and $p_3 = \frac{1}{2}$.

IMM - DTU

02405 Probability
2006-2-2
BFN/bfn

Solution for exercise 1.4.10 in Pitman

We define the events

$S1$ Source one works

$S2$ Source two works

$W0$ No working sources

$W1$ One source works

$W2$ Two sources work

E Enough power awailable.

Question a) The event $W0 = S1^c \cap S2^c$. Since $S1$ and $S2$ are independent we know that $S1^c$ and $S2^c$ are independent too. Thus

$$P(W0) = P(S1^c \cap S2^c) = P(S1^c)P(S2^c) = (1-P((S1))(1-P(S2)) = (1-0.4)(1-0.5) = 0.3$$

$$P(W2) = P(S1 \cap S2) = P(S1)P(S2) = 0.4 \cdot 0.5 = 0.2$$

The three events $W0$, $W1$, and $W2$ constitute a partition of the outcome space (p. 20). Thus

$$P(W1) = 1 - P(W0) - P(W2) = 0.5$$

Question b) Once again using that $W0$, $W1$, and $W2$ constitute a partition and use the rule of averaged conditional probability p. 41 (p. 73)

$$P(E) = P(E|W0)P(W0)+P(E|W1)P(W1)+P(E|W2)P(W2) = 0 \cdot 0.3+0.6 \cdot 0.5+1 \cdot 0.2 = 0.5$$

IMM - DTU

02405 Probability
2006-9-13
BFN/bfn

Solution for exercise 1.5.1 in Pitman

We introduce the events

O The odd box is picked

B A black marble is picked

We have $P(O) = \frac{1}{2}$ and $P(B|O) = \frac{1}{4}, P(B|O^c) = \frac{2}{6} = \frac{1}{3}$.

Question a) The events O and O^c is a partition (page 20, see also page 40). We apply the rule of averaged conditional probabilities (box at top of page 41, summary page 73) to get

$$P(B) = P(B|O)P(O) + P(B|O^c)P(O^c) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{6} = \frac{7}{24}$$

Question b) The probability in question is $P(O^c|B^c)$,which is a standard setting for the application of Bayes rule (interchange of conditioning) page 49 or summary page 73. We get

$$P(O^c|W) = \frac{P(W|O^c)P(O^c)}{P(W)} = \frac{\frac{4}{6} \cdot \frac{1}{2}}{\frac{17}{24}} = \frac{8}{17}$$

Remark: We could have written the denominator in full as on page 49. However the denominator is simply the probability of the event which we condition upon on the left side of Bayes rule; that is the event A in the genereal form page 49, or in this case the event B.

$\square$

IMM - DTU

02405 Probability
2003-9-11
BFN/bfn

Solution for exercise 1.5.3 in Pitman

C The event that the chip is ok

A The event that a chip is accepted by the cheap test

Question a)

$$P(C|A) = \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|C^c)P(C)^c} = \frac{1 \cdot 0.8}{0.8 + 0.1 \cdot 0.2}$$

Question b) We introduce the event

S Chip sold

$$P(S) = 0.8 + 0.2 \cdot 0.1 = 0.82$$

The probability in question is

$$P(C^c|S) = \frac{P(S|C^c)P(C^c)}{P(S|C^c)P(C^c) + P(S|C)P(C)} = \frac{0.1 \cdot 0.2}{0.02 + 1 \cdot 0.8} = \frac{1}{41}$$

IMM - DTU

02405 Probability
2003-9-13
BFN/bfn

Solution for exercise 1.5.5 in Pitman

Define the events

H A randomly selected person is healthy

D A randomly selected person is diagnosed with the disease

Question a) From the text we have the following quantities

$$P(H) = 0.99 \qquad P(D|H) = 0.05 \qquad P(D|H^c) = 0.8$$

and from the law of averaged conditional probabilities we get

$$P(D) = P(H)P(D|H) + P(H^c)P(D|H^c) = 0.99 \cdot 0.05 + 0.01 \cdot 0.8 = 0.0575$$

Question b) The proability in question

$$P(H^c \cap D^c) = P(H^c)P(D^c|H^c) = 0.01 * 0.2 = 0.002$$

using the multiplication (chain) rule

Question c) The proability in question

$$P(H \cap D^c) = P(H)P(D^c|H) = 0.99 * 0.95 = 0.9405$$

using the multiplication (chain) rule

Question d) The probability in question is $P(H^c|D)$. We use Bayes rule to "interchange" the conditioning

$$P(H^c|D) = \frac{P(D|H^c)P(H^c)}{P(D|H^c)P(H^c) + P(D|H)P(H)} = 0.8 \cdot 0.010.008 + 0.05 \cdot 0.99 = 0.139$$

Question e) The probabilities are estimated as the percentage of a large group of people, which is indeed the frequency interpretation.

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 1.5.9 in Pitman

Denote the event that a shape of type i is picked by T_i, the event that it lands flat by F and the event that the number rolled is six by S. We have $P(T_i) = \frac{1}{3}, i = 1, 2, 3$, $P(F|T_1) = \frac{1}{3}, P(F|T_2) = \frac{1}{2}$, and $P(F|T_3) = \frac{2}{3}$ $P(S|F) = \frac{1}{2}$, and $P(S|F^c) = 0$.

Question a) We first note that the six events $T_i \cap F$ and $T_i \cap F^c$ $(i = 1, 2, 3)$ is a partition of the outcome space. Now using The Rule of Averaged Conditional Probabilities (The Law of Total Probability) page 41

$$P(S) = P(S|T_1 \cap F)P(T_1 \cap F) + P(S|T_2 \cap F)P(T_2 \cap F) + P(S|T_3 \cap F)P(T_3 \cap F) + P(S|T_1 \cap F^c)P(T_1 \cap F^c) + \ldots$$

The last three terms are zero. We apply The Multiplication Rule for the probabilities $P(T_i \cap F)$ leading to

$$P(S) = P(S|T_1 \cap F)P(F|T_1)P(T_1) + P(S|T_2 \cap F)P(F|T_2)P(T_2) + P(S|T_3 \cap F)P(F|T_3)P(T_3)$$

a special case of The Multiplication Rule for n Events page 56. Inserting numbers

$$P(S) = \frac{1}{2}\frac{1}{3}\frac{1}{3} + \frac{1}{2}\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{1}{4}$$

Question b) The probability in question is $P(T_1|S)$. Applying Bayes' rule page 49

$$P(T_1|S) = \frac{P(S|T_1)P(T_1)}{P(S)} = \frac{\frac{1}{6}\frac{1}{3}}{\frac{1}{4}} = \frac{2}{9}$$

IMM - DTU

02405 Probability
2004-2-7
BFN/bfn

Solution for exercise 1.6.1 in Pitman

This is another version of the birthday problem. We denote the event that the first n persons are born under different signs, exactly as in example 5 page 62. Correspondingly, R_n denotes the event that the n'th person is the first person born under the same sign as one of the previous $n-1$ persons. We find

$$P(D_n) = \prod_{i=1}^{n} \left(1 - \frac{i-1}{12}\right), \qquad n \leq 13$$

We find $P(D_4) = 0.57$ and $P(D_5) = 0.38$.

IMM - DTU

02405 Probability
2003-9-24
BFN/bfn

Solution for exercise 1.6.5 in Pitman

Question a) We will calculate the complementary probability, the no student has the same birthday and do this sequentially. The probability that the first student has a different birthday is $\frac{364}{365}$, the same is true for all the remaining $n-2$ students. The probability in question is

$$P(\text{All other } n-1 \text{ students has a different birthday than no.1}) = 1 - \left(\frac{364}{365}\right)^{n-1}$$

Question b)

$$1 - \left(\frac{364}{365}\right)^{n-1} \geq \frac{1}{2} \Leftrightarrow n \geq \frac{\ln(2)}{\ln(365) - \ln(364)} + 1 = 253.7$$

Question c) In the birthday problem we only ask for two arbitrary birthdays to be the same, while the question in this exercise is that at least one out of $n-1$ has a certain birthday.

IMM - DTU

02405 Probability
2003-9-18
BFN/bfn

Solution for exercise 1.6.6 in Pitman

Question a) By considering a sequence of throws we get

$$\begin{aligned}
P(1) &= 0 \\
P(2) &= \frac{1}{6} \\
P(3) &= \frac{5}{6}\frac{2}{6} \\
P(4) &= \frac{5}{6}\frac{4}{6}\frac{3}{6} \\
P(5) &= \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{4}{6} \\
P(6) &= \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{5}{6} \\
P(7) &= \frac{5}{6}\frac{4}{6}\frac{3}{6}\frac{2}{6}\frac{1}{6}
\end{aligned}$$

Question b) The sum of the probabilities p_2 to p_6 must be one, thus the sum in question is 1.

Question c) Can be seen immediately.

IMM - DTU

02405 Probability
2004-9-24
BFN/bfn

Solution for exercise 1.6.7 in Pitman

Question a) The exercise is closely related to example 7 p.68. Using the same notation and approach

$$P(\text{Current flows}) = P((S_1 \cup S_2) \cap S_3) = (1 - P(S_1^c \cap S_2^c))P(S_3) = (1 - q_1 q_2)q_3$$

(use $1 = p_1p_2 + q_1p_2 + p_1q_2 + q_1q_2$ to get the result in Pitman)

Question b)

$$P(\text{Current flows}) = P(((S_1 \cup S_2) \cap S_3) cup S_4) = 1 - (1 - q_1 q_2)q_3 q_4$$

(or use exclusion/inclusion like Pitman)

IMM - DTU

02405 Probability
2003-9-13
BFN/bfn

Solution for exercise 1.6.8 in Pitman

question a) The events B_{ij} occur with probability

$$P(B_{ij}) = \frac{1}{365}$$

It is immediately clear that

$$P(B_{12} \cap B_{23}) = \frac{1}{365^2} = P(B_{12})P(B_{23}).$$

implying independence. The following is a formal and lengthy argument. Define A_{i_j} as the the event that the i'th person is born the j'th day of the year.

We have $P(A_{i_j}) = \frac{1}{365}$ and that A_{1_i}, $A_{2,j}$, $A_{3,k}$, and $A_{4,l}$ are independent. The event B_{ij} can be expressed by

$$B_{ij} = \cup_{k=1}^{365} (A_{i,k} \cap A_{j,k})$$

such that $P(B_{ij}) = \frac{1}{365}$ by the independence of $A_{i,k}$ and $A_{j,k}$. The event $B_{12} \cap B_{23}$ can be expressed by

$$B_{12} \cap B_{23} = \cup_{k=1}^{365} (A_{1,k} \cap A_{2,k} \cap A_{3,k})$$

and by the independence of the A's we get $P(B_{12} \cap B_{23}) = \frac{1}{365^2}$

question b) The probability

$$P(B_{13}|B_{12} \cap B_{23}) = 1 \neq P(B_{13})$$

thus, the events B_{12}, B_{13}, B_{23} are not independent.

question c) Pairwise independence follows from a)

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 2.1.1 in Pitman

Question a) We use the formula for the number of combinations - appendix 1, page 512 (the binomial coefficient)

$$\binom{7}{4} = \binom{7}{3} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Question b) The probability in question is given by the binomial distribution, see eg. page 81.

$$35 \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)^4 = \frac{35 \cdot 125}{6^7} = 0.0156$$

IMM - DTU

02405 Probability
2003-9-13
BFN/bfn

Solution for exercise 2.1.2 in Pitman

We define the events Gi: i girls in family. The probabilities $P(Gi)$ is given by the binomial distribution due to the assumptions that the probabilities that each child is a girl do not change with the number or sexes of previous children.

$$P(Gi) = \binom{4}{i} \frac{1}{2}^i \frac{1}{2}^{4-1}, \qquad P(G2) = 6 \cdot \frac{1}{16} = \frac{3}{8}$$

$$P(G2^c) = 1 - P(G2) = \frac{5}{8}$$

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 2.1.4 in Pitman

We denote the event that there are 3 sixes in 8 rolls by A, the event that there are 2 sixes in the first 5 rolls by B. The probability in question is $P(B|A)$. Using the general formula for conditional probabilities page 36

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

The probability $P(B \cap A) = P(A|B)P(B)$ by the multiplication rule, thus as a speical case of Bayes Rule page 49 we get

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Now the probability of $P(A)$ is given by the binomial distribution page 81, as is $P(B)$ and $P(A|B)$ (the latter is the probability of getting 1 six in 3 rolls). Finally

$$P(B|A) = \frac{P(\text{2 sixes in 5 rolls})P(\text{1 six in 3 rolls})}{P(\text{3 sixes in 8 rolls})} = \frac{\binom{5}{2}\frac{5^3}{6^5}\binom{3}{1}\frac{5^2}{6^3}}{\binom{5}{2}\frac{5^5}{6^8}} = \frac{\binom{5}{2}\binom{3}{1}}{\binom{8}{3}}$$

a hypergeometric probability. The result generalizes. If we have x successes in n trials then the probability of having $y \leq x$ successes in $m \leq n$ trials is given by

$$\frac{\binom{m}{y}\binom{n-m}{x-y}}{\binom{n}{x}}$$

The probabilities do not depend on p.

IMM - DTU

02405 Probability
2005-5-30
BFN/bfn

Solution for exercise 2.1.6 in Pitman

We define events Bi that the man hits the bull's eye exactly i times. The probabilities of the events Bi is given by the Binomial distribution

$$P(Bi) = \binom{8}{i} 0.7^i 0.3^{8-i}$$

Question a) The probability of the event

$$P(B4) = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} 0.7^4 0.3^4 = 0.1361$$

Question b)

$$P(B4 | \cup_{i=2}^{8} B_i) = \frac{P\left((B4 \cap (\cup_{i=2}^{8} B_i)\right)}{P\left(\cup_{i=2}^{8} B_i\right)} = \frac{P(B4)}{1 - P(B0) - P(B1)} == 0.1363$$

Question c)

$$\binom{6}{2} 0.7^2 0.3^4 = 0.0595$$

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 2.2.1 in Pitman

All questions are answered by applying The Normal Approximation to the Binomial Distribution page 99 (131). We have $\mu = n \cdot p = 400 \cdot \frac{1}{2} = 200,\ \sigma = \sqrt{npq} = \sqrt{400\frac{1}{2}\frac{1}{2}} = 10$. The questions differ only in the choice of a and b in the formula.

Question a) $a = 190, b = 210$

$$P(190 \text{ to } 210 \text{ successes}) = \Phi\left(\frac{210.5 - 200}{10}\right) - \Phi\left(\frac{189.5 - 200}{10}\right)$$

$$= \Phi(1.05) - \Phi(-1.05) = 0.8531 - (1 - 0.8531)0.7062$$

Question b) $a = 210, b = 220$

$$P(210 \text{ to } 220 \text{ successes}) = \Phi\left(\frac{220.5 - 200}{10}\right) - \Phi\left(\frac{209.5 - 200}{10}\right)$$

$$= \Phi(2.05) - \Phi(0.95) = 0.9798 - 0.8289 = 0.1509$$

Question c) $a = 200, b = 200$

$$P(200 \text{ successes}) = \Phi\left(\frac{200.5 - 200}{10}\right) - \Phi\left(\frac{199.5 - 200}{10}\right)$$

$$= \Phi(0.05) - \Phi(-0.05) = 0.5199 - (1 - 0.5199) = 0.0398$$

Question d) $a = 210, b = 210$

$$P(210 \text{ successes}) = \Phi\left(\frac{210.5 - 200}{10}\right) - \Phi\left(\frac{209.5 - 200}{10}\right)$$

$$= \Phi(1.05) - \Phi(0.95) = 0.8531 - 0.8289 = 0.0242$$

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 2.2.4 in Pitman

We apply The Normal Approximation to the Binomial Distribution page 99. Note that $b\tilde{=}\infty$ such that the first term is 1. We have $\mu = n \cdot p = 300 \cdot \frac{1}{3} = 100$ and $\sigma = \sqrt{300\frac{1}{3}\frac{2}{3}} = 10\sqrt{\frac{2}{3}}$. The value of a in the formula is 121 (more than 120). We get

$$P(\text{More than 120 patients helped} = 1-\Phi\left(\frac{120.5-100}{8.165}\right) = 1-\Phi(2.51) = 1-0.994 = 0.006$$

IMM - DTU

02405 Probability
2004-5-16
BFN/bfn

Solution for exercise 2.2.6 in Pitman

We introduce the events O_i to describe that i voters in the survey oppose the measure. From section 2.1 box at bottom of page 81 we deduce that X is binomially distributed.

Question a) The probability in question is (page 81)

$$P(O_{90}) = \binom{200}{90} 0.45^{90} 0.55^{110}$$

We evaluate this probability by approximation page 99.

$$P(O_{90}) \simeq \Phi\left(\frac{90+\frac{1}{2}-0.45\cdot 200}{\sqrt{200\cdot 0.45\cdot(1-0.45)}}\right) - \Phi\left(\frac{90-\frac{1}{2}-0.45\cdot 200}{\sqrt{200\cdot 0.45\cdot(1-0.45)}}\right)$$

$$= \Phi(0.07) - \Phi(-0.07) = 2\cdot\Phi(0.07) - 1 = 0.056$$

Question b) The probability in question is

$$P(\text{more than 100 voters oppose the measure}) = P\left(\cup_{i=101}^{200} O_i\right) \simeq$$

$$\Phi\left(\frac{200+\frac{1}{2}-0.45\cdot 200}{\sqrt{200\cdot 0.45\cdot(1-0.45)}}\right) - \Phi\left(\frac{101-\frac{1}{2}-0.45\cdot 200}{\sqrt{200\cdot 0.45\cdot(1-0.45)}}\right) = 1-\Phi(1.49) = 0.0681$$

IMM - DTU

02405 Probability
2006-9-29
BFN/bfn

Solution for exercise 2.2.9 in Pitman

We define the events S_i that i passengers show up. The probability of the event S_i is given by the Binomial distribution, and can be approximated using the normal approximation

Question a)

$$P(\text{More than 300 passengers show up}) = 1 - P(\text{At most 300 passengers show up}) =$$

$$1 - \Phi\left(\frac{300 + \frac{1}{2} - 0.9 \cdot 324}{\sqrt{324 \cdot 0.1 \cdot 0.9}}\right) = 1 - \Phi(1.65) = 0.0495$$

Question b) Increase; the relative variability increases.

Question c)

$$P(\text{More than 150 pairs show up}) = 1 - \Phi\left(\frac{150 + \frac{1}{2} - 0.9 \cdot 162}{\sqrt{162 \cdot 0.1 \cdot 0.9}}\right) = 1 - \Phi(1.23) = 0.1093$$

IMM - DTU

02405 Probability
2003-9-19
BFN/bfn

Solution for exercise 2.2.14 in Pitman

Question a) We define the events Wi that a box contains i working devices. The probability in question can be established by

$$-P(W390 \cup W391 \cup W392 \cup W393 \cup W394 \cup W395 \cup W396 \cup W397 \cup W398 \cup W399 \cup W400)$$

$$= P(W390)+P(W391)+P(W392)+P(W393)+P(W394)+P(W395)+P(W396)+P(W397)+P(W39$$

since the event Wi are mutually exclusive. The probabilities $P(Wi)$ are given by the binomial distribution

$$P(i) = \binom{400}{i} 0.95^i 0.05^{400-i},$$

we prefer to use the normal approximation, which is

$$1-P(\text{less than 390 working}) \simeq 1-\Phi\left(\frac{390-\frac{1}{2}-400\cdot 0.95}{\sqrt{400\cdot 0.95\cdot 0.05}}\right) = 1-\Phi(2.18) = 1-0.9854 = 0.0146$$

Without continuity correction we get $1-\Phi(2.29) = 0.0110$ The skewness correction is:

$$-\frac{1}{6}\frac{1-2\cdot 0.95}{\sqrt{400\cdot 0.95\cdot 0.95}}(2.18^2-1)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}2.18^2} = 0.0048$$

The skewness correction is quite significant and should be applied. Finally we approximate the probability in question with 0.00098, which is still somewhat different from the exact value of 0.0092.

Question b)

$$P(\text{at least } k) \simeq 1-\Phi\left(\frac{k+\frac{1}{2}-400\cdot 0.95}{\sqrt{400\cdot 0.95\cdot 0.05}}\right) \geq 0.95$$

With

$$\frac{k+\frac{1}{2}-400\cdot 0.95}{\sqrt{400\cdot 0.95\cdot 0.05}} \leq -1.645$$

we find $k = 373$.

IMM - DTU

02405 Probability
2003-9-24
BFN/bfn

Solution for exercise 2.4.7 in Pitman

Question a) From page 90 top we know that m is the largest integer less than equal to $(n+1) \cdot p = 2.6$, thus $m = 2$.

Question b)

$$\binom{25}{2} 0.1^2 0.9^{23} = 0.2659$$

Question c)

$$\Phi\left(\frac{2+\frac{1}{2}-2.5}{\sqrt{25 \cdot 0.09}}\right) - \Phi\left(\frac{1+\frac{1}{2}-2.5}{\sqrt{25 \cdot 0.09}}\right) = \Phi(0) - \Phi(-0.667) = 0.2475$$

Question d)

$$\frac{2.5^2}{2!} \cdot e^{-2.5} = 0.2566$$

Question e) Normal m is now 250

$$\Phi\left(\frac{250+\frac{1}{2}-250}{\sqrt{2500 \cdot 0.09}}\right) - \Phi\left(\frac{250-\frac{1}{2}-250}{\sqrt{2500 \cdot 0.09}}\right) = \Phi(\frac{1}{30}) - \Phi(-\frac{1}{30}) = 0.0266$$

Question f) Poisson - as above 0.2566.

IMM - DTU

02405 Probability
2003-9-19
BFN/bfn

Solution for exercise 2.4.8 in Pitman

The Poisson probabilities $P_\mu(k)$ are

$$P_\mu(k) = \frac{\mu^k}{k!}e^-\mu$$

We use odds ratio for the probabilities

$$\frac{P(k+1)}{P(k)} = \frac{\frac{\mu^{k+1}}{(k+1)!}e^-\mu}{\frac{\mu^k}{k!}e^-\mu} = \frac{\mu}{k+1}$$

The ratio is strictly decreasing in k. For $\mu < 1$ maximum will be $P_\mu(0)$, otherwise the probabilities will increase for all k such that $\mu > k$, and decrease whenever $\mu < k$. For non-integer μ the maximum of $P_\mu(k)$ (the mode of the distribution) is obtained for the largest $k < \mu$. For μ intger the value of $P_\mu(\mu) = P_\mu(\mu + 1)$.

IMM - DTU

02405 Probability
2003-9-13
BFN/bfn

Solution for exercise 2.4.10 in Pitman

The probability of the event that there is at least one success can be calculated using the Binomial distribution. The probability of the complentary event that there is no successes in n trials can be evaluated by the Poisson approximation.

$$P(0) = e^{-\frac{1}{N}\frac{2}{3}N} = 0.5134$$

Similarly for $n = \frac{5}{3}N$

$$P(0) + P(1) = e^{-\frac{1}{N}\frac{5}{3}N}\left(1 + \frac{5}{3}\right) = 0.5037$$

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for exercise 2.5.1 in Pitman

Question a) We use the hypergeometric distribution page 125 since we are dealing with sampling without replacement

$$P(\text{Exactly 4 red tickets}) = \frac{\binom{20}{4}\binom{30}{6}}{\binom{50}{10}}$$

Question b) We apply the binomial distribution (sampling with replacement page 123)

$$P(\text{Exactly 4 red tickets}) = \binom{10}{4}\left(\frac{20}{50}\right)^4\left(\frac{30}{50}\right)^6 = 210\frac{2^4 3^6}{5^{10}}$$

Solution for exercise 2.5.4 in Pitman

We have sampling without replacement. The probability in question can be derived from the result on page 125. First we use this result to state the probability that we get exactly i men in the sample

$$P(i) = P(i \text{ men}) = \frac{\binom{40,000}{i}\binom{60,000}{100-i}}{\binom{100,000}{100}}$$

then the probability in question can be found as

$$P(\text{at least 45 men in sample}) = \sum_{i=45}^{100} \frac{\binom{40,000}{i}\binom{60,000}{100-i}}{\binom{100,000}{100}}.$$

We approximate the probabilities $P(i)$ using the binomial distribution

$$P(i) \tilde{=} \binom{100}{i} 0.4^i 0.6^{100-i}$$

$$P(\text{at least 45 men in sample}) = \sum_{i=45}^{100} \binom{100}{i} 0.4^i 0.6^{100-i} = 1 - P(\text{at most 44 men in sample}).$$

The latter probability can be evaluated approximately with the normal approximation.

$$P(\text{at most 44 men in sample}) \tilde{=} \Phi\left(\frac{44+\frac{1}{2}-40}{\sqrt{100 \cdot 0.4 \cdot 0.6}}\right) = \Phi(0.92) = 0.8212.$$

Finally

$$P(\text{at least 45 men in sample}) \tilde{=} 0.1788$$

(the skewness correction is (0.0003) if you would like to apply that too).

IMM - DTU

02405 Probability
2005-2-20
BFN/bfn

Solution for exercise 2.5.5 in Pitman

The probability in question is given by the Binomial distribution evaluated with the Normal approximation (boxed result page 99). Let A_i define the event that i voters in the sample prefer A. Then $P(A_i)$ is given by the $Bin(n, 0.55)$ distribution. We want to determine n such that $P\left(\cup_{i>\frac{n}{2}} A_i\right) \geq 0.99 \Leftrightarrow P\left(\cup_{i\leq\frac{n}{2}} A_i\right) \leq 0.01$. Expressed differently $P\left(0 \leq \text{Number preferring } B \leq \frac{n}{2}\right)$.

$$\Phi\left(\frac{\frac{n}{2}+\frac{1}{2}-0.55\cdot n}{\sqrt{n\cdot 0.55\cdot 0.45}}\right) \leq 0.99$$

Thus

$$\frac{\frac{n}{2}+\frac{1}{2}-0.55\cdot n}{\sqrt{n\cdot 0.55\cdot 0.45}} \leq -2.33 \Rightarrow n > 557 \quad .$$

Pitman gets 537 ignoring the continuity approximation.

IMM - DTU

02405 Probability
2003-10-3
BFN/bfn

Solution for exercise 2.5.9 in Pitman

Question a) The probability that the second sample is drawn is the probability that the first sample contains exactly one bad item, which occurs with probability

$$p_1 = \frac{\binom{10}{1}\binom{40}{4}}{\binom{50}{5}}$$

(the hypergeometric distribution page 125). The probability that the second sample contains more than one bad item is calculated via the probability of the complementary event, i.e. that the second sample contains one or two bad items, which is

$$p_2 = \frac{\binom{9}{0}\binom{36}{10}}{\binom{45}{10}} + \frac{\binom{9}{1}\binom{36}{9}}{\binom{45}{10}}$$

The answer to the question is the product of these two probabilities $p_1(1-p_2) = 0.2804$.

Question b) The lot is accepted if we have no bad items in the first sample or the event described under a)

$$\frac{\binom{10}{0}\binom{40}{5}}{\binom{50}{5}} + \frac{\binom{10}{1}\binom{40}{4}}{\binom{50}{5}}\left(\frac{\binom{9}{0}\binom{36}{10}}{\binom{45}{10}} + \frac{\binom{9}{1}\binom{36}{9}}{\binom{45}{10}}\right) = 0.4595$$

Solution for exercise 3.1.1 in Pitman

Question a) The probabilities $P(X=i), i=0,1,2,3$ are given by the binomial distribution, $P(X=i)=\binom{3}{i}\frac{1}{2^3}$.

i	$P(X=i)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

Question b) We define a random variable $Y=|X-1|$, with range 0,1,2.

Then

i	$P(Y=i)$
0	$\frac{3}{8}$
1	$\frac{1}{2}$
2	$\frac{1}{8}$

IMM - DTU

02405 Probability
2003-9-21
BFN/bfn

Solution for exercise 3.1.4 in Pitman

Question a)

$X2/X1$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

Question b)

$Y2/Y1$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
2	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
3	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
4	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$
6	0	0	0	0	0	$\frac{1}{36}$

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.1.5 in Pitman

The random variable $Z = X_1 X_2$ has range $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 36\}$. We find the probability of $Z = i$ by counting the combinations of X_1, X_2 for which $X_1 X_2 = i$. we get:

$Z = i$	$P(Z = i)$
1	$\frac{1}{36}$
2	$\frac{2}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{2}{36}$
6	$\frac{4}{36}$
8	$\frac{2}{36}$
9	$\frac{1}{36}$
10	$\frac{2}{36}$
12	$\frac{4}{36}$
15	$\frac{2}{36}$
16	$\frac{1}{36}$
18	$\frac{2}{36}$
20	$\frac{2}{36}$
24	$\frac{2}{36}$
25	$\frac{1}{36}$
30	$\frac{2}{36}$
36	$\frac{1}{36}$

Solution for exercise 3.1.12 in Pitman

Question a) Binomial $B(n, p_i)$

Question b) Binomial $B(n, p_i + p_j)$

Question c) Multinomial $P(n, p_i, p_j, 1 - p_i - p_j)$

IMM - DTU

02405 Probability
2003-11-30
BFN/bfn

Solution for exercise 3.1.14 in Pitman

Question a) We define the events Gg as the the events that team A wins in g games. The probabilities $P(Gg)$ can be found by thinking of the game series as a sequence of Bernoulli experiments. The event Gg is the event that the fourth succes (win by team A) occurs at game g. These probabiliites are given by the negative binomial distribution (page 213 or page 482). Using the notation of the distribution summary page 482, we identify $r = 4$, $n = g - 4$ (i.e. counting only the games that team A loses). We get

$$P(Gg) = \binom{g-1}{4-1} p^4 q^{g-4} \qquad g = 4, 5, 6, 7$$

Question b)

$$p^4 \sum_{g=4}^{7} \binom{g-1}{3} q^{g-4}$$

Question c) The easiest way is first answering question d) then using $1 - \mathit{binocdf}(3, 7, 2/3)$ in MATLAB.

$$0.8267$$

Question d) Imagine that all games are played etc. From the binomial formula

$$p^7 + 7p^6q + 21p^5q^2 + 35p^4q^3 = p^7 + p^6q + 6p^6q + 6p^5q^2 + 15p^5q^2 + 35p^4q^3$$

$$= p^6 + 6p^5q + 15p^4q^2 + 20p^4q^3 = p^6 + p^5q + 5p^5q + 15p^4q^2 + 20p^4q^3$$

etc.

Question e)

$$P(G = 4) = p^4 + q^4 \qquad P(G = 5) = 4pq(p^3 + q^3)$$

$$P(G = 6) = 10p^2q^2(p^2 + q^2) \qquad P(G = 7) = 20p^3q^3(p + q)$$

Independence for $p = q = \frac{1}{2}$

IMM - DTU

02405 Probability
2003-11-21
BFN/bfn

Solution for exercise 3.1.16 in Pitman

Question a) Using the law of averaged conditional probabilities we get

$$P(X+Y=n) = \sum_{i=0}^{n} P(X=i)P(X+Y=n|X=i) = \sum_{i=0}^{n} P(X=i)P(Y=n-i)$$

where the last equality is due to the independence of X and Y.

Question b) The marginal distribution of X and Y is

$$P(X=2)=\frac{1}{36}, \qquad P(X=3)=\frac{1}{18}, \qquad P(X=4)=\frac{1}{12}$$

$$P(X=5)=\frac{1}{9}, \qquad P(X=6)=\frac{5}{36}, \qquad P(X=7)=\frac{1}{6}$$

We get

$$P(X+Y=8) = 2\left(\cdot\frac{1}{36}\cdot\frac{5}{36}+\frac{1}{18}\frac{1}{9}\right)+\frac{1}{12}\cdot\frac{1}{12} = \frac{35}{16\cdot 81}$$

IMM - DTU

02405 Probability
2003-10-6
BFN/bfn

Solution for exercise 3.1.24 in Pitman

Question a) We define $P(X \text{ even}) = P(Y \text{ even}) = p$, and introduce the random variable $W = X + Y$. The probability p_w of the event that W is even is

$$p_w = p^2 + (1-p)(1-p) = 2p^2 + 1 - 2p = (1-p)^2 + p^2$$

with minimum $\frac{1}{2}$ for $p = \frac{1}{2}$.

Question b) We introduce $p_0 = P(X \mod 3 = 0), p_1 = P(X \mod 3 = 1), p_2 = P(X \mod 3 = 2)$. The probability in question is

$$p_0^3 + p_1^3 + p_2^3 + 3p_0p_1p_2$$

which after some manipulations can be written as

$$1 - (p_0p_1 + p_0p_2 + p_1p_2 - 3p_0p_1p_2)$$

The expressions can be maximized/minimized using standard methods, I haven't found a more elegant solution than that.

Solution for exercise 3.2.1 in Pitman

$$15 \cdot 0.1 + 25 \cdot 0.2 + 50 \cdot 0.7 = 41.5$$

IMM - DTU

02405 Probability
2003-10-1
BFN/bfn

Solution for exercise 3.2.2 in Pitman

Let us denote the numbers on the first list by x_i and the numbers on the second list by y_i The average of the first list is $1 \cdot 0.2 + 2 \cdot 0.8 = 1.8$. The average of the second list is $3 \cdot 0.5 + 5 \cdot 0.5 = 4.0$.

Question a) The average of the list made by addition is 5.8. This can be seen by

$$\frac{1}{100}\sum_{i=1}^{100}(x_i + y_i) = \frac{1}{100}\sum_{i=1}^{100} x_i + \frac{1}{100}\sum_{i=1}^{100} y_i$$

Question b) The average of the list made by subtraction is -2.2 by the same approach.

Question c) The average in question is

$$\frac{1}{100}\sum_{i=1}^{100} x_i y_i$$

and we need some information on the ordering to calculate the sum, thus we do not have sufficient information.

Question d) As c).

IMM - DTU

02405 Probability
2004-5-13
BFN/bfn

Solution for exercise 3.2.3 in Pitman

Question a) Let X define the number of sixes appearing on three rolls. We find $P(X=0) = \left(\frac{5}{6}\right)^3$, $P(X=1) = 3\frac{5^2}{6^3}$, $P(X=2) = 3\frac{5}{6^3}$, and $P(X=3) = \frac{1}{6^3}$. Using the definition of expectation page 163

$$E(X) = \sum_{x=0}^{3} x¶(X = x) = 0 \cdot \left(\frac{5}{6}\right)^3 + 1 \cdot 3\frac{5^2}{6^3} + 2 \cdot 3\frac{5}{6^3} + 3 \cdot \frac{1}{6^3} = \frac{1}{2}$$

or realizing that $X \in \textit{binomial}\left(3, \frac{1}{6}\right)$ example 7 page 169 we have $E(X) = 3 \cdot \frac{1}{6} = \frac{1}{2}$.

Question b) Let Y denote the number of odd numbers on three rolls, then $Y \in \textit{binomial}\left(3, \frac{1}{2}\right)$ thus $E(Y) = 3 \cdot \frac{1}{2} = \frac{3}{2}$.

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.2.7 in Pitman

We define the indicator variables I_i which are 1 of switch i are closed 0 elsewhere. We have $X = I_1 + I_2 + \cdots + I_n$, such that

$$E(X) = E(I_1 + I_2 + \cdots + I_n) = E(I_1) + E(I_2) + \cdots + E(I_n) = p_1 + p_2 + \cdots + p_n = \sum_{i=0}^{n} p_i$$

Solution for exercise 3.2.8 in Pitman

$E((X+Y)^2) = E(X^2+Y^2+2XY) = E(X^2)+E(Y^2)+2E(XY) = 11$

IMM - DTU

02405 Probability
2005-3-10
BFN/bfn

Solution for exercise 3.2.14 in Pitman

The event Bi that at least one person gets off at floor i. Using indicators I_{B_i} we introduce the random variable N as the number of stops. We have

$$N = I_{B_1} + \cdots + I_{B_10} \qquad E(N) = E(I_{B_1} + \cdots + I_{B_10})$$

$E(N) = E(I_{B_1} + \cdots + I_{B_10}) \qquad = E(I_{B_1}) + \cdots + E(I_{B_10}) = P(B_1) + \cdots + P(B_10) = 10P(B_1)$

We find $P(B_1) = 1 - P(B_1^c) = 1 - \left(\frac{9}{10}\right)^{12}$ thus $E(N) = 10\left(1 - \left(\frac{9}{10}\right)^{12}\right) = 7.18$

IMM - DTU

02405 Probability
2003-10-12
BFN/bfn

Solution for exercise 3.2.17 in Pitman

Question a) The event $D \leq 9$ occurs if all the red balls are among the first 9 balls drawn. The probability of this event is given by the Hypergeometric distribution p. 125 and 127.

$$P(D \leq 9) = \frac{\binom{3}{3}\binom{10}{6}}{\binom{13}{9}} = 0.2937$$

Question b)

$$P(D = 9) = P(D \leq 9) - P(D \leq 8) = \frac{\binom{3}{3}\binom{10}{6}}{\binom{13}{9}} - \frac{\binom{3}{3}\binom{10}{5}}{\binom{13}{8}} = 0.2284$$

Question c) To calculate the mean we need the probabilities of $P(D = i)$ for $i = 3, 4, \ldots, 13$. We get

$$P(D \leq i) = \frac{\binom{3}{3}\binom{10}{i-3}}{\binom{13}{i}} = \frac{\binom{10}{i-3}}{\binom{13}{i}} = \frac{\frac{10!}{(13-i)!(i-3)!}}{\frac{13!}{(13-i)!i!}} = \frac{10!i!}{13!(i-3)!} = \frac{i(i-1)(i-2)}{13 \cdot 12 \cdot 11}$$

$$P(D = i) = P(D \leq i) - P(D \leq i-1) = \frac{i(i-1)(i-2)}{13 \cdot 12 \cdot 11} - \frac{(i-1)(i-2)(i-3)}{13 \cdot 12 \cdot 11} = \frac{3(i-1)(i-2)}{13 \cdot 12 \cdot 11}$$

$$E(D) = \sum_{i=3}^{12} i \frac{3(i-1)(i-2)}{13 \cdot 12 \cdot 11} = \frac{3}{13 \cdot 12 \cdot 11} \sum_{i=3}^{12} i(i-1)(i-2) = \frac{3}{13 \cdot 12 \cdot 11} 6,006 = 10.5$$

IMM - DTU

02405 Probability
2003-10-12
BFN/bfn

Solution for exercise 3.2.21 in Pitman

Question a) We have by definition of the indicator that

$$I_{A^c} = \begin{cases} 1 & \text{if } A^c \text{ occurs} \\ 0 & \text{if } (A^c)^c \text{ occurs} \end{cases} = \begin{cases} 1 & \text{if } A^c \text{ occurs} \\ 0 & \text{if } A \text{ occurs} \end{cases}$$

which is $1 - I_A$.

Question b)

$$I_{AB} = \begin{cases} 1 & \text{if } AB \text{ occurs} \\ 0 & \text{if } (AB)^c \text{ occurs} \end{cases}$$

etc.

Question c) By rewriting $I_{A_1 \cup A_2 \cup \cdots \cup A_n}$ as $1 - I_{(A_1 \cup A_2 \cup \cdots \cup A_n)^c} = 1 - I_{(A_1^c \cap A_2^c \cap \cdots \cap A_n^c)}$ and applying the results of a) and b).

Question d) By expanding the product and taking expectation on both sides, then using $E(I_A) = P(A)$.

IMM - DTU

02405 Probability
2005-3-11
BFN/bfn

Solution for exercise 3.3.1 in Pitman

Question a)

# of days	28	30	31
frequency	1	4	7

$$E(X) = 28 \cdot \frac{1}{12} + 30 \cdot \frac{4}{12} + 31 \cdot \frac{7}{12} = \frac{365}{12} = (30.42)$$
$$E(X^2) = 28^2 \cdot \frac{1}{12} + 30^2 \cdot \frac{4}{12} + 31^2 \cdot \frac{7}{12}$$
$$SD(X) = \sqrt{E(X^2) - E(X)^2} = 0.86$$

Question b)

# of days	28	30	31
frequency	$\frac{28}{365}$	$\frac{120}{365}$	$\frac{217}{365}$

$$E(X) = 28 \cdot \frac{28}{365} + 30 \cdot \frac{120}{365} + 31 \cdot \frac{217}{365} = \frac{11111}{365}$$
$$E(X^2) = 28^2 \cdot \frac{28}{365} + 30^2 \cdot \frac{120}{365} + 31^2 \cdot \frac{217}{365} = \frac{338489}{365}$$
$$SD(X) = \sqrt{E(X^2) - E(X)^2} = 0.841$$

IMM - DTU

02405 Probability
2005-10-29
BFN/bfn

Solution for exercise 3.3.2 in Pitman

The random variable Y is binominally distributed. The mean $E(Y) = 3\frac{1}{2} = \frac{3}{2}$, the variance is $V(Y) = 3\frac{1}{2}\frac{1}{2} = \frac{3}{4}$. The Mean of Y^2 is

$$E(Y^2) = \sum_{i=0}^{3} i^2 \left(\begin{array}{c} 3 \\ i \end{array} \right) \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{3-i} = 1\frac{3}{8} + 4\frac{3}{8} + 9\frac{1}{8} = 3$$

Alternatively we could have used the computational formula for the variance

$$V(Y) + (E(Y))^2 = \frac{3}{4} + \frac{9}{4} = 3$$

$$P(Y=0) = \left(\frac{1}{2}\right)^3 \qquad P(Y=1) = 3\left(\frac{1}{2}\right)^3 \qquad P(Y=2) = 3\left(\frac{1}{2}\right)^3 \qquad P(Y=3) = \left(\frac{1}{2}\right)^3$$

The variance of Y^2 can be found by

$$V(Y^2) = E((Y^2)^2) - (E(Y^2))^2$$

We need to calculate $E((Y^2)^2) = E(Y^4)$

$$E(Y^4) = \frac{3 + 16 \cdot 3 + 81 \cdot 1}{8} = \frac{132}{8} = \frac{33}{2}$$

Finally

$$V(Y^2) = \frac{33}{2} - 9 = \frac{15}{2}$$

IMM - DTU

02405 Probability
2003-10-2
BFN/bfn

Solution for exercise 3.3.4 in Pitman

The computational formula for the variance page 186 is quite useful (important). This exercise is solved by applying it twice. First we use it once to get:

$$Var(X_1X_2) = E((X_1X_2)^2) - (E(X_1X_2))^2$$

Now by the independence of X_1 and X_2

$$E((X_1X_2)^2)-(E(X_1X_2))^2 = E(X_1^2X_2^2)-(E(X_1)E(X_2))^2 = E(X_1^2)E(X_2^2)-(E(X_1)E(X_2))^2$$

using the multiplication rule for Expectation page.177 valid for independent random variables. We have also used the fact that if X_1 and X_2 are independent then $f(X_1)$ and $g(X_2)$ are independent too, for arbitrary functions $f()$ and $g()$. We now use the computational formula for the variance once more to get

$$Var(X_1X_2) = (Var(X_1) + (E(X_1))^2)(Var(X_2) + (E(X_2))^2) - (E(X_1)E(X_2))^2$$

Now inserting the symbols of the exercise we get

$$Var(X_1X_2) = \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2$$

IMM - DTU

02405 Probability
2003-10-2
BFN/bfn

Solution for exercise 3.3.14 in Pitman

Question a) Markov's inequality

$$P(X \geq 50,000) \leq \frac{E(X)}{50,000} = \frac{1}{5}$$

Question b) Chebychevs inequality

$$P(|X - E(X)| \geq kSD(X)) = \leq \frac{1}{k^2}$$

we have $k = 5$ such that the probability is bounded by $\frac{1}{25}$. The bound provided by Chebychevs inequality is much sharper than the one provided by Markov's inequality.

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.3.19 in Pitman

We apply the Normal approximation (the Central Limit Theorem (p.196). Let X_i denote the weight of the i'th passenger. The total load W is $W = \sum_{i=1}^{30} X_i$.

$$P(W > 5000) \tilde{=} 1 - \Phi\left(\frac{5000 - 30 \cdot 150}{55\sqrt{30}}\right) = 1 - \Phi(1.66) = 0.0485$$

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.3.23 in Pitman

We define S_n as the time of installment of the n'th battery. Similarly we define N_t to be the number of batteries replaced in the interval $[0,t($. We have $P(S_n \leq t) = P(N_t \geq n)$, thus $P(N_{104} \geq 26) = P(S_{26} \leq 104)$ where the time unit is weeks. We now apply the Normal approximation (Central Limit Theorem) to S_{26}.

$$P(S_{26} \leq 104) \tilde{=} \Phi\left(\frac{104 - 26 \cdot 4}{1 \cdot \sqrt{104}}\right) = 0.5$$

Solution for exercise 3.4.1 in Pitman

Question a) X the number of heads in 9 tosses is binominally distributed, thus

$$P(X = 5) = \binom{9}{5} p^5 (1-p)^4$$

Question b) Y the number of tosses for the first head is geometrically distributed, thus

$$P(Y = 7) = (1-p)^6 p$$

Question c) Z the number of tosses to get 5 heads follows a negative binomial distribution

$$P(Z = 12) = \binom{11}{4} (1-p)^7 p^5$$

Question d) X_1 the number of heads in the first 8 tosses and X_2 the number of heads in the next 5 tosses are independent. We get

$$\sum_{i=0}^{5} \binom{8}{i} p^i (1-p)^{8-i} \binom{5}{i} p^i (1-p)^{5-i} = \sum_{i=0}^{5} \binom{8}{i} \binom{5}{i} p^{2i} (1-p)^{13-2i}$$

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.4.2 in Pitman

First we restate D : number of balls drawn to get two of the same colour. We draw one ball which is either red or black. Having drawn a ball of some colour the number of draws to get one of the same colour is geometrically distributed with probability $\frac{1}{2}$. Thus $D = X + 1$ where X is geometrically distributed with $p = \frac{1}{2}$.

Question a)

$$P(D = i) = p(1-p)^{i-2}, \qquad p = 2, 3, \ldots$$

Question b)

$$E(D) = E(X+1) = E(X) + 1 = \frac{1}{p} + 1 = 3$$

from page 212 or 476,482.

Question c)

$$V(D) = V(X+1) = V(X) = \frac{1-p}{p^2} = 2, \qquad SD(D) = \sqrt{2}$$

from page 213 or 476,482.

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 3.4.9 in Pitman

We define the random variable N as the number of throws to get heads. The pay back value is N^2, the expected win from the game can be expressed as

$$E(N^2 - 10) = E(N^2) - 10$$

using the rule for the expectation of a linear function of a random variable p. 175 b. We could derive $E(N^2)$ from the general rule for expectation of a function of a random variable p. 175 t. However, it is more convenient to use the fact the N follows a Geometric distribution and use the Computational Formula for the Variance p. 186.

$$E(N^2) = Var(N) + (E(N))^2 = \frac{1-p}{p^2} + \left(\frac{1}{p}\right)^2 = 2 + 4 = 6$$

The values for $Var(N)$ and $E(N)$ can be found p. 476 in the distribution summary.

IMM - DTU

02405 Probability
2006-3-10
BFN/bfn

Solution for exercise 3.4.12 in Pitman

We will use the formula for the geometric series

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 \ldots = \frac{1}{1-q}, \qquad |q| < 1$$

repeatedly in this exercise.

Question a) The Rule of Averaged Conditional Probabilities p.41 applied for a countable rather than a finite partitioning. See also p.209 Infinite Sum Rule.

$$P(W_1 = W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 = W_1 | W_1 = w)$$

Now using the indpendence of W_1 and W_2 we get

$$P(W_1 = W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 = w) = \sum_{w=1}^{\infty} p_1 (1-p_1)^{w-1} p_2 (1-p_2)^{w-1}$$

$$= p_1 p_2 \sum_{k=0}^{\infty} ((1-p_1)(1-p_2))^k = \frac{p_1 p_2}{1-(1-p_1)(1-p_2))}$$

Question b) Similarly

$$P(W_1 < W_2) = \sum_{w=1}^{\infty} P(W_1 = w) P(W_2 > w) = \sum_{w=1}^{\infty} p_1 (1-p_1)^{w-1} (1-p_2)^w = \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2))}$$

where the relation $P(W_2 > w) = (1-p_2)^w$ can be derived using the formula of the geometric series. The result is stated directly page 482.

Question c)

$$P(W_2 < W_1) = \frac{p_2(1-p_1)}{1-(1-p_1)(1-p_2))}$$

Question d)

$$P(\min(W_1, W_2) = w) = P(W_1 = W_2 = w) + P(W_1 = w, W_2 > w) + P(W_1 > w, W_2 = w)$$

$$= p_1 p_2 ((1-p_1)(1-p_2))^{w-1} + p_1 (1-p_1)^{w-1} (1-p_2)^w + (1-p_1)^w p_2 (1-p_2)^{w-1}$$

$$= (p_1 p_2 + p_1(1-p_2) + (1-p_1) p_2)((1-p_1)(1-p_2))^{w-1} = (1-(1-p_1)(1-p_2))((1-p_1)(1-p_2))^{w-1}$$

We could have stated this result directly without calculations, by considering two simultaneously running series of Bernoulli experiments, where we stop as soon as we get a succes in one of them.

Question e)

$$P(\max(W_1, W_2) = w) = P(W_1 = W_2 = w) + P(W_1 = w, W_2 < w) + P(W_1 < w, W_2 = w)$$

$$= p_1 p_2 ((1-p_1)(1-p_2))^{w-1} + p_1 (1-p_1)^{w-1} (1-(1-p_2)^{w-1}) + (1-(1-p_1)^{w-1}) p_2 (1-p_2)^{w-1}$$

$$= p_1(1-p_1)^{w-1} + p_2(1-p_2)^{w-1} - (p_1 + p_2 - p_1 p_2)((1-p_1)(1-p_2))^{w-1}$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 3.4.17 in Pitman

We introduce N as the number of children and D as the number of boys in a family. The number of boys in a family of size n is *bin* $\left(n, \frac{1}{2}\right)$ distributed. By applying the rule of averaged conditional probabilities we get

$$P(D=k) = \sum_{n=k}^{\infty} P(D=k|N=n)P(N=n) = \sum_{n=k}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{2}\right)^n \cdot p^n(1-p)$$

The terms in the sum are close to the terms of a negative binomial distribution distribution see summary page 482 or derivation page 213 Example 4. We first identify the parameter r to be k+1. The probability of a succes is $\left(1-\frac{p}{2}\right)$. Summing over all possible outcomes $t \geq r$) for an $NB\left(k+1, \left(1-\frac{p}{2}\right)\right)$ distribution (using the distribution in the standard form -page 215 or page 482) gives

$$\sum_{m=0}^{\infty} \left(\begin{array}{c} m+(k+1)-1 \\ (k+1)-1 \end{array} \right) \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^m = 1.$$

Now using an appropriate change of summation variable ($n = m + k$)

$$\sum_{n=k}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^{n-k} = 1$$

We now need to manipulate our expression for $P(D=k)$ to apply this result thus eliminating or evaluating the sum.

$$P(D=k) = \sum_{n=k}^{\infty} P(D=k|N=n)P(N=n) = \sum_{n=k}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\frac{1}{2}\right)^n \cdot p^n(1-p)$$

$$= \frac{(1-p)\left(\frac{p}{2}\right)^k}{\left(1-\frac{p}{2}\right)^{k+1}} \sum_{n=k}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) \left(1-\frac{p}{2}\right)^{k+1} \left(\frac{p}{2}\right)^{n-k} = \frac{2(1-p)p^k}{(2-p)^{k+1}}$$

Solution for exercise 3.5.4 in Pitman

We define the stochastic variables X_i as the number of misprints on page i. We assume that the number of characters on each page are approximately the same and that misprints occur independently of each other with a fixed probability for each character. We will evaluate probabilities using the Poisson distribution.

$$P(X_i < 5) = P(X_i = 0) + P(X_i = 1) + P(X_i = 2) + P(X_i = 3) + P(X_i = 4)$$

$$= e^{-1}(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}) = \frac{65}{24}e^{-1} = 0.9963$$

The event that at least one page has at least 5 misprints is complementary to the event that all pages has at most 4 misprints.

$$P(\max(X)_i) \geq 5) = 1 - P(X_i \leq 4)^{300} = 0.6671$$

IMM - DTU

02405 Probability
2005-10-29
BFN/bfn

Solution for exercise 3.5.5 in Pitman

Assuming the microbes are randomly distributed we apply the Poisson distribution. The parameter of the Poisson distribution is found using the Poisson Scatter Theorem p. 230 t., thus $5,000 \cdot 10^{-4} = 0.5$. Applying this we get

$$P(\text{at least one microbe}) = 1 - e^{-0.5} = 0.3935$$

IMM - DTU

02405 Probability
2006-11-1
BFN/bfn

Solution for exercise 3.5.9 in Pitman

Question a)

$$P(X=1,Y=2)=P(X=1)P(Y=2)=\frac{1^1}{1!}e^{-1}\frac{2^2}{2!}e^{-2}=2e^{-3}$$

Question b)

$$P\left(\frac{X+Y}{2}\geq 1\right)=P(X+Y\geq 2)=1-P(X+Y\leq 1)$$

$$=1-(P(X+Y=0)+P(X+Y=1))=1-(1+3)e^{-3}=0.80,$$

where we use a) to find $P(X+Y=0)$ and $P(X+Y=1)$.

Question c)

$$P\left(X=1|\frac{X+Y}{2}=2\right)=P\left(X=1|X+Y=4\right)=\frac{P(X=1,X+Y=4)}{P(X+Y=4)}$$

$$=\frac{e^{-1}\frac{2^3}{3!}e^{-2}}{\frac{3^4}{4!}e^{-3}}=\binom{4}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3=0.395$$

the conditional probability is given by the Binomial distribution. This is a general result.

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.5.10 in Pitman

Question a) X is Poisson distributed with parameter λ. Using page 175, linear functions of X,

$$E(3X+5) = 3E(X)+5$$

and the mean of a Poisson distributed random variable page 223

$$E(3X+5) = 3E(X)+5 = 3\lambda+5$$

Question b) Using linear functions of X page 188, - here called Scaling and Shifting, and the variance of a Poisson distributed random variable page 223.

$$V(3X+5) = 9V(X) = 9\lambda$$

Question c) We use the definition of the expectation of a function of a random variable p.175

$$\sum_{i=0}^{\infty} \frac{1}{i+1}\frac{\lambda^i}{i!}e^{-\lambda} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{(i+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda}\sum_{i=0}^{\infty}\frac{\lambda^{i+1}}{(i+1)!}$$

Now

$$\sum_{i=0}^{\infty}\frac{a^i}{i!} = e^a$$

such that

$$\frac{e^{-\lambda}}{\lambda}\sum_{i=0}^{\infty}\frac{\lambda^{i+1}}{(i+1)!} = \frac{1}{\lambda}\left(1-e^{-\lambda}\right)$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 3.5.13 in Pitman

Question a) Using the Poisson Scatter Theorem p.230 we get

$$\mu(x) = x^3 \frac{6.023 \cdot 10^{23}}{22.4 \cdot 10^3} = 2.688 \cdot 10^{19} x^3$$

and

$$\sigma(x) = \sqrt{\mu(x)} = 5.1854 \cdot 10^9 x \sqrt{x}$$

Question b)

$$\frac{5.1854 \cdot 10^9 x \sqrt{x}}{2.688 \cdot 10^{19} x^3} \geq 0.01 \rightarrow x \leq 7.1914 \cdot 10^{-6}$$

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.5.16 in Pitman

We assume that the chocolate chips and mashmallows are randomly scattered in the dough.

Question a) The number of chocoloate chips in one cubic inch is Poisson distributed with parameter 2 according to our assumptions. The number of chocolate chips in thre cubic inches is thus Poisson distributed with parameter 6. Let X denote the number of chocolate chops in a three cubic inch cookie.

$$P(X \leq 4) = e^{-6}\left(1 + 6 + \frac{36}{2} + \frac{36 \cdot 6}{6} + \frac{216 \cdot 6}{4 \cdot 6}\right) = 115 \cdot e^{-6} = 0.285$$

Question b) We have three Poisson variates X_i : total number of chocolate chips and marshmallows in cookie i. According to our assumptions, X_1 follows a Poisson distribution with parameter 6, while X_2 and X_3 follow a Poisson distribution with parameter 9. The complementary event is the event that we get two or three cookies without chocoloate chips and marshmallows.

$$P(X_1 = 0, X_2 = 0, X_3 = 0) + P(X_1 > 1, X_2 = 0, X_3 = 0)$$

$$+P(X_1 = 0, X_2 > 1, X_3 = 0) + P(X_1 = 0, X_2 = 0, X_3 > 1)$$
$$= e^{-6}e^{-9}e^{-9} + (1 - e^{-6})e^{-9}e^{-9} + e^{-6}(1 - e^{-9})e^{-9} + e^{-6}e^{-9}(1 - e^{-9}) \simeq 0$$

we are almost certain that we will get at most one cookie without goodies.

IMM - DTU

02405 Probability
2003-10-5
BFN/bfn

Solution for exercise 3.5.18 in Pitman

Question a) The variable X_1 is the sum of a thinned Poisson variable (X_0) and a Poisson distributed random variable (the immigration). The two contributions are independent, thus X_1 is Poisson distributed. The same argument is true for any n and we have proved that X_n is Poisson distributed by induction. Ee denote the parameter of the n'th distribution by λ_n. We have the following recursion:

$$\lambda_n == p\lambda_{n-1} + \mu$$

with $\lambda_0 = \mu$ such that

$$\lambda_1 = (1+p)\mu$$

and more generally

$$\lambda_n = \sum_{i=0}^{n} p^i \mu = \mu \frac{1-p^{n+1}}{1-p}$$

Question b) As $n \to \infty$ we get $\lambda_n \to \frac{\mu}{1-p}$. This value is also a fixpoint of

$$\lambda_n == p\lambda_{n-1} + \mu$$

IMM - DTU

02405 Probability
2004-3-3
BFN/bfn

Solution for exercise 3.5.2 in Pitman

Assuming the number of raisins in a cookie X can be described by a Poisson distribution and that the mean value of X is λ we get

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$$

solving

$$e^{-\lambda} = 0.01$$

gives

$$\lambda = 4.605$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 4.1.1 in Pitman

Question a) We apply the result for the infinitesimal probability page 263, and recall the standard normal density page 266

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$P(X \in [0, 0.001]) = f(0) \cdot 0.001 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}0^2} \cdot 0.001 = 3.99 \cdot 10^{-4}$$

Question b) We follow the same approach as in a)

$$P(X \in [1, 1.001]) = f(1) \cdot 0.001 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}1^2} \cdot 0.001 = 2.42 \cdot 10^{-4}$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 4.1.2 in Pitman

Question a) The integral of $f()$ should be one for $f()$ to be a density (p.263).

$$\int_1^\infty \frac{c}{x^4}\mathrm{d}x = [-\frac{c}{3}\frac{1}{x^3}]_{x=1}^{x=\infty} = \frac{c}{3}$$

We conclude $c = 3$.

Question b) Using the definition of $E(g(X))$ page 263, we find the mean of X to be

$$\int_1^\infty xf(x)\mathrm{d}x = \int_1^\infty x\frac{3}{x^4}\mathrm{d}x = \frac{3}{2}$$

Question c) The Computational Formula for Variance is still valid (page 261). We get

$$E(X^2) = \int_1^\infty x^2 f(x)\mathrm{d}x = \int_1^\infty x^2\frac{3}{x^4}\mathrm{d}x = 3, \qquad Var(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

IMM - DTU

02405 Probability
2003-10-15
BFN/bfn

Solution for exercise 4.1.4 in Pitman

Question a) The integral of $f(x)$ over the range of X should be one (see e.g. page 263).

$$\int_0^1 x^2(1-x)^2 \mathrm{d}x = \int_0^1 x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x$$

using the binomial formula $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$.

$$\int_0^1 x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = \sum_{i=0}^{2} \binom{2}{i} \int_0^1 (-x)^{i+2} \mathrm{d}x = \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+3}}{i+3} \right]_{x=0}^{x=1} = \frac{1}{30}$$

such that

$$f(x) = 30 \cdot x^2(1-x)^2 \qquad 0 < x < 1$$

This is an example of the Beta distribution page 327,328,478.

Question b) We derive the mean

$$\int_0^1 x f(x) \mathrm{d}x = \int_0^1 x 30 \cdot x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = 30 \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+4}}{i+4} \right]_{x=0}^{x=1} = \frac{1}{2}$$

which we could have stated directly due to the symmetry of $f(x)$ around $\frac{1}{2}$, or from page 478.

Question c) We apply the computational formula for variances as restated page 261.

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^1 x^2 30 \cdot x^2 \left(\sum_{i=0}^{2} \binom{2}{i} (-x)^i \right) \mathrm{d}x = 30 \sum_{i=0}^{2} \binom{2}{i} (-1)^i \left[\frac{x^{i+5}}{i+5} \right]_{x=0}^{x=1} = \frac{30}{105}$$

such that

$$Var(X) = \frac{30}{105} - \frac{1}{4} = \frac{1}{28}$$

which can be verified page 478.

$$SD(X_{3,3})^2 = \frac{3 \cdot 3}{(3+3)^2(3+3+1)} = \frac{1}{28}$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 4.1.5 in Pitman

Question a)

Question b) We apply the formula on page 263 for a density

$$P(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x$$

We get

$$P(-1 \leq X \leq 2) = \int_{-1}^{2} \frac{1}{2(1+|x|)^2}\mathrm{d}x = \int_{-1}^{0} \frac{1}{2(1-x)^2}\mathrm{d}x + \int_{0}^{2} \frac{1}{2(1+x)^2}\mathrm{d}x$$

$$= \left[\frac{1}{2(1-x)}\right]_{x=-1}^{x=0} + \left[-\frac{1}{2(1+x)}\right]_{x=0}^{x=2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{6} = \frac{7}{12}$$

Question c) The distribution is symmetric so $P(|X| > 1) = 2P(X > 1) = 2\left[-\frac{1}{2(1+x)}\right]_{x=1}^{x=\infty} = \frac{1}{2}$.

Question d) No. (the integral $\int_0^\infty x\frac{1}{2(1+x)^2}\mathrm{d}x$ does not exist).

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 4.1.9 in Pitman

We first determine S_4 and $Var(S_4)$. From the distribution summary page 477 we have $E(S_4) = 4\frac{1}{2} = 2$ and due to the independence of the X_i's we have $Var(S_4) = 4\frac{1}{12} = \frac{1}{3}$. (the result from the variance follows from the result page 249 for a sum of independent random variables and the remarks page 261 which states the validity for continuous distributions). We now have

$$P(S_4 \geq 3) = 1 - \Phi\left(\frac{3-2}{\sqrt{\frac{1}{3}}}\right) = 1 - \Phi(1.73) = 1 - 0.9582 = 0.0418$$

Solution for exercise 4.1.12 in Pitman

Question a) First we determine the total area of the figure, which is 8. The area of the triangle with x-coordinate less than or equal to x_0 is $\frac{1}{2}(x+2)\cdot 2(x+2) = (x+2)^2$ for $x \le 0$ and $8-\frac{1}{2}(2-x)2(2-x) = 4+4x-x^2$ for $0 < x \le 2$, such that $P(X \le x) = \frac{(x+2)^2}{8}$ for $x \le 0$. To find the density $f(x)$ we consider $P(x < X \le x+\Delta x)$ for Δx small. We get

$$P(x < X \le x+\Delta x) = \frac{((x+\Delta x)+2)^2}{8} - \frac{(x+2)^2}{8} = \frac{x+2}{4}\Delta x + \frac{(\Delta x)^2}{8} \quad .$$

We now find

$$f(x) = \begin{cases} \frac{x+2}{4} & -2 \le x \le 0 \\ \frac{2-x}{4} & 0 < x \le 2 \end{cases}$$

Question b) We can write the density as

$$f(x) = \begin{cases} c(2+x) & -2 \le x < 0 \\ 2c(1-x) & 0 \le x \le 1 \end{cases}$$

From integration (or by considering the area of the figure) we find $c = \frac{1}{3}$.

Question c) The four lines defining the square are: $y = 2x - \frac{1}{2}, y = 2 - \frac{1}{2}x, y = 2x+2$ and $y = -\frac{1}{2}x - \frac{1}{2}$. The area of the square is 5. Now considering $P(x < X < x + \mathrm{d}x)$ for $-1 < x < 0$. The triangle defined by the vertical line through $x, y = 2x+2$ and $y = -\frac{1}{2}x - \frac{1}{2}$ has area $\frac{1}{2}x(2x+2--\frac{1}{2}x-\frac{1}{2} = \frac{1}{4}(x+1)^2$. We find the area of the triangle defined by the vertical line through $x+\mathrm{d}x, y = 2x+2$ and $y = -\frac{1}{2}x - \frac{1}{2}$ to $\frac{1}{4}(x+\mathrm{d}x+1)^2$ and derive $f(x)\mathrm{d}x = \frac{1}{2}(x+1)$. By using similar arguments for the intervals $(0,1)$ and $(1,2)$ we get

$$f(x) = \begin{cases} \frac{1}{2}(1+x) & -1 \le x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{2}(2-x) & 1 \le x \le 2 \end{cases}$$

IMM - DTU

02405 Probability
2003-11-10
BFN/bfn

Solution for exercise 4.1.13 in Pitman

Question a) We derive the density of the distribution

$$f(x) = \begin{cases} c(x-0.9) & 0.9 < x \le 1.0 \\ c(1-x) & 1.0 < x < 1.1 \end{cases}$$

We can find c the standard way using $\int_{0.9}^{1.1} f(x)\mathrm{d}x = 1$. However, we can derive the area of the triangle directly as $\frac{1}{2} \cdot 0.02 \cdot c$ such that $c = 100$. Due to the symmetry of $f(x)$ we have $P(X < 0.925) = P(1.075 < X)$.

$$P(\text{rod scrapped}) = 2P(X < 0.925) = 2\int_{0.9}^{0.925} 10(x-0.9)\mathrm{d}x = 20\left[\frac{1}{2}x^2 - 0.9x\right]_{x=0.9}^{x=0.925} = 0.0625$$

Question b) We define the random variable Y as the length of an item which has passed the quality inspection. The probability

$$P(0.95 < Y < 1.05) = \frac{P(0.95 < X < 1.05)}{P(0.925 < X < 1.075)} = \frac{0.75}{0.9375} = 0.8$$

The number of acceptable items A out of c are binomially distributed. We determine c such that

$$P(A \ge 100) \ge 0.95$$

We now use the normal approximation to get

$$1 - \Phi\left(\frac{100 - 0.5 - 0.8 \cdot c}{0.4\sqrt{c}}\right) \ge 0.95$$

$$\frac{100 - 0.5 - 0.8 \cdot c}{0.4\sqrt{c}} \le -1.645$$

and we find $c \ge 134$.

IMM - DTU

02405 Probability
2005-10-29
BFN/bfn

Solution for exercise 4.2.1 in Pitman

We use the knowledge of the half-life to find λ from example 2 page 282, $\lambda = \frac{\ln(2)}{1}$

Question a) The probability that an atom survives at least 5 years is given by the survival function page 279. We get with T denoting the life time of an atom

$$P(T > 5) = e^{-5\ln(2)} = \frac{1}{32}$$

Question b) If we let N_t denote the number of atoms surviving at time t, then the distribution of this random variable will be binomial with the probability found in the previous question. Thus $N_t \in bin\left(n, e^{-t\ln(2)}\right)$, where $n = N_0$ is the original number of atoms. The expected value $E(N_t)$ of this binomial distribution is given page 476 or 479 in the distribution summary, such that $E(N_t) = ne^{-t\ln(2)}$. We find $t_{10\%}$ as , using the method on page 282 once more,

$$ne^{-t\ln(2)} = \frac{n}{10} \Rightarrow t_{10\%} = \frac{-\ln(0.1)}{\ln(2)}.$$

Question c) Applying the same method to find the time where the expected number of atoms remaining of 1024 is 10 ($t^\star$), we get

$$1024e^{-t^\star\ln(2)} = 1 \Rightarrow t^\star = \frac{\ln(1024)}{\ln(2)} = 10$$

Question d) This question can be formulated as $P(N_{t^\star} = 0)$. From the binomial distribution of $N_{t^\star}$ we get

$$\left(\frac{1023}{1024}\right)^{1024} = 0.3677$$

or

$$\left(\frac{1023}{1024}\right)^{1024} = \left(1 - \frac{1}{1024}\right)^{1024} \simeq e^{-1}$$

IMM - DTU

02405 Probability
2003-10-23
BFN/bfn

Solution for exercise 4.2.4 in Pitman

Question a) We define T_i as the lifetime of component i. The probability in question is given by the Exponential Survival Function p.279. The mean is $10 hours$, thus $\lambda = 0.1h^{-1}$.

$$P(T_i > 20) = e^{-0.1 \cdot 20} = e^{-2} = 0.1353$$

Question b) The problem is similar to the determination of the *half life* of a radioactive isotope Example 2. p.281-282. We repeat the derivation

$$P(T_i \leq t_{50\%}) = 0.5 \Leftrightarrow e^{-\lambda t_{50\%}} = 0.5 \qquad t_{50\%} = \frac{\ln 2}{\lambda} = 6.93$$

Question c) We find the standard deviation directly from page 279

$$SD(T_i) = \frac{1}{\lambda} = 10$$

Question d) The average life time $\bar{T}$ of 100 components is

$$\bar{T} = \frac{1}{100} \sum_{i=1}^{100} T_i$$

We know from page 286 that $\bar{T}$ is Gamma distributed. However, it is more convenient to apply CLT (Central Limit Theorem) p.268 to get

$$P(\bar{T} > 11) = 1 - P(\bar{T} \leq 11) \tilde{=} 1 - \Phi\left(\frac{11 - 10}{\frac{10}{\sqrt{100}}}\right) = 1 - \Phi(1) = 0.1587$$

Question e) The sum of the lifetime of two components is Gamma distributed. From p.286 (Right tail probability) we get

$$P(T_1 + T_2 > 22) = e^{-0.1 \cdot 22}(1 + 2.2) = 0.3546$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for review exercise 1 (chapter 1) in Pitman

Solution for exercise 4.2.5 in Pitman

Question a) The time between two calls in a Poisson process is exponentially distributed (page 289). Using the notation of page 289 with $\lambda = 1$ we get

$$P(W_4 \leq 2) = 1 - e^{-2} = 0.8647$$

Question b) The distribution of the time to the arrival of the fourth call is a Gamma $(4, \lambda)$ distribution. We find the probability using the result (2) on page 286

$$P(T_4 \leq 5) = 1 - e^{-5}\left(1 + 5 + \frac{25}{2} + \frac{125}{6}\right) = 1 - \frac{118}{3}e^{-5} = 0.735$$

Question c)

$$E(T_4) = \frac{4}{\lambda} = 4$$

using (3) page 286.

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for exercise 4.2.8 in Pitman

We introduce the events M_i that the transistor is produced on machine i.

Question a) Using the Rule of Average Conditional Probabilities page 41 we get

$$P(X \geq 200) = P(X \geq 200|M_1)P(M_1) + P(X \geq 200|M_2)P(M_2) = e^{-\frac{200}{100}}\frac{4}{12} + e^{-\frac{200}{200}}\frac{8}{12} = 0.2904$$

Question b) Similarly

$$E(X) = 100 \cdot \frac{1}{3} + 200 \cdot \frac{2}{3} = \frac{500}{3}$$

Question c) To find the variance we use the Computational Formula for Variance page 261. We introduce X_i to be the lifetime of a transistor produced by machine i. We use the Computational Formula inversely to get

$$E(X_i^2) = V(X_i) + (E(X_i))^2 = \frac{2}{\lambda_i^2}$$

where $E(X_i) = \frac{1}{\lambda_i}$ is the mean lifetime of a transistor produced on machine i.

$$E(X^2) = E(X_1^2)\frac{1}{3} + E(X_2^2)\frac{2}{3} = 6 \cdot 100^2$$

Finally

$$Var(X) = 6 \cdot 100^2 - \left(\frac{500}{3}\right)^2 = \frac{29}{9} \cdot 100^2$$

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.2.9 in Pitman

Question a)

$$\Gamma(r+1) = \int_0^\infty x^r e^{-x} \mathrm{d}x = \left[x^r(-e^{-x})\right]_0^\infty - \int_0^\infty r x^{r-1}(-e^{-x})\mathrm{d}x = r\int_0^\infty r x^{r-1} e^{-x}\mathrm{d}x = r\Gamma(r)$$

Question b) For $r = 1$ we have

$$\Gamma(1) = \int_0^\infty e^{-x}\mathrm{d}x = 1$$

and the result is proved by induction.

Question c)

$$E(T^n) = \int_0^\infty t^n f(t)\mathrm{d}t = \int_0^\infty t^n e^{-t}\mathrm{d}t = \Gamma(n+1)$$

$$Var(T) = E(T^2) - (E(T))^2 = \Gamma(3) - (\Gamma(2))^2 = 2 - 1 = 1$$

Question d) We introduce the random variable $Y = \lambda T$. The survival function of Y $G_Y(y)$ can be derived through

$$G_Y(y) = P(Y > y) = P(\lambda T > y) = P\left(T > \frac{y}{\lambda}\right)$$

Now $P(T > x)e^{-\lambda x}$ such that

$$G_Y(y) = e^{-\lambda\frac{y}{\lambda}} = e^{-y}$$

the survival function of an *exponential*(1) variable. Now

$$E(T^n) = \frac{1}{\lambda^n}E((\lambda T)^n) = \frac{n!}{\lambda^n}$$

since $E((\lambda T)^n) = n!$ (the variable $Y = \lambda T$ is an *exponential*(1) distributed random variable).

IMM - DTU

02405 Probability
2003-10-30
BFN/bfn

Solution for exercise 4.2.10 in Pitman

Question a) We define $T_1 = \int (T)$ such that

$$P(T_1 = 0) = 1 - P(T > 1) = 1 - e^{-\lambda}$$

using the survival function for an exponential random variable. Correspondingly

$$P(K = k) = P(T > k) - P(T > k+1) = e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}\left(1 - e^{-\lambda}\right) = \left(e^{-\lambda}\right)^k\left(1 - e^{-\lambda}\right)$$

a geometric distribution with parameter $p = 1 - e^{-\lambda}$.

Question b)

$$P(T_m = k) = P(T > \frac{k}{m}) - P(T > \frac{k+1}{m}) = e^{-\lambda\frac{k}{m}} - e^{-\lambda\frac{k+1}{m}} = \left(e^{-\frac{\lambda}{m}}\right)^k\left(1 - e^{-\frac{\lambda}{m}}\right)$$

$p_m = e^{-\frac{\lambda}{m}}$.

Question c) The mean of the geometric distribution of T_m is

$$E(T_m) = \frac{1 - p_m}{p_m}$$

The mean is measured in $\frac{1}{m}$ time units so we have to multiply with this fraction to get an approximate value for $E(T)$

$$E(T) \tilde{=} = \frac{1}{m}E(T_m) = \frac{1 - p_m}{p_m}$$

$$= \frac{1}{m}\frac{e^{-\frac{\lambda}{m}}}{1 - e^{-\frac{\lambda}{m}}} = \frac{1}{m}\frac{1 - \frac{\lambda}{m} + o\left(\frac{\lambda}{m}\right)}{1 - \left(1 - \frac{\lambda}{m} + o\left(\frac{\lambda}{m}\right)\right)} \rightarrow \frac{1}{\lambda} \text{ for } m \rightarrow infty$$

Solution for exercise 4.3.1 in Pitman

Question a) The survival function $G(t) = P(T > t)$ is introduced and defined page 297

$$P(T \leq b) = 1 - P(T > b) = 1 - G(b)$$

Question b)

$$P(a \leq T \leq b) = P(T \leq b) - P(T < a) = G(a) - G(b)$$

($P(T < a) = P(T \leq a)$ for a continuous distribution).

Solution for exercise 4.3.2 in Pitman

We first show that a constant Hazard rate implies an exponential distribution. Using (7) page 297

$$G(t) = e^{\int_0^t \lambda \mathrm{d}u} = e^{-\lambda t}$$

the survival function of an exponential distribution. The density of an exponential distribution with parameter(intensity) λ is $f(t) = \lambda e^{-\lambda t}$. The hazard rate is found using (6) page 297

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

and the proof is complete.

Solution for exercise 4.3.3 in Pitman

Question a) We find $G(t)$ using (7) page 297

$$G(t) = e^{-\int_0^t \frac{a}{b+u}\mathrm{d}u} = e^{-a[\ln(b+u)]_{u=0}^{u=t}} = e^{-a\ln\left(\frac{b+t}{b}\right)} = \left(1+\frac{t}{b}\right)^{-a}$$

This is a Pareto distribution. The Pareto distribution is one of the generic distributions with important applications in economics (income distributions), insurance (claim size distribution), geology (distribution for strenght of earth quakes), and telecommunications (duration of internet connections).

Question b) We find $f(t)$ using (5) page 297

$$f(t) = -\frac{\mathrm{d}G(t)}{\mathrm{d}t} = -\frac{\mathrm{d}\left(1+\frac{t}{b}\right)^{-a}}{\mathrm{d}t} = \frac{a}{b}\left(1+\frac{t}{b}\right)^{-a-1}$$

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.3.4 in Pitman

The relation between the hazard rate $\lambda(t)$ and the survival function $G(t)$ is given by (7) page 297

$$G(t) = e^{-\int_0^t \lambda(u) \mathrm{d}u}$$

Now inserting $\lambda(u) = \lambda\alpha u^{\alpha-1}$

$$G(t) = e^{-\int_0^t \lambda\alpha u^{\alpha-1} \mathrm{d}u} = e^{-\lambda[u^\alpha]_{u=0}^{u=t}} = e^{-\lambda t^\alpha}$$

Similarly we derive $f(t)$ from $G(t)$ using (5) page 297

$$f(t) = -\frac{\mathrm{d}G(t)}{\mathrm{d}t} = -e^{-\lambda t^\alpha}\left(-\lambda\alpha t^{\alpha-1}\right) = \lambda\alpha t^{\alpha-1} e^{-\lambda t^\alpha}$$

Finally from (6) page 297

$$\lambda(t) = \frac{\lambda\alpha t^{\alpha-1} e^{-\lambda t^\alpha}}{e^{-\lambda t^\alpha}} = \lambda\alpha t^{\alpha-1}$$

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.3.6 in Pitman

The hazard rate is given by

$$\lambda(t) = \begin{cases} 0.05 & t \leq 10 \\ 0.1 & t > 10 \end{cases}$$

Question a) Using the relation (7) page 297 we get

$$G(t) = e^{-(10 \cdot 0.05 + 5 \cdot 0.1)} = e^{-1}$$

Question b)

$$G(t) = \begin{cases} e^{-0.05t} & t \leq 10 \\ e^{-0.5} e^{-0.1(t-10)} & t > 10 \end{cases}$$

Question c) Using (5) page 297 we get

$$f(t) = \begin{cases} 0.05 e^{-0.05t} & t \leq 10 \\ 0.1 e^{-0.5} e^{-0.1(t-10)} & t > 10 \end{cases}$$

Question d) We calculate the mean using (8) page 299.

$$E(T) = \int_0^\infty G(t) \mathrm{d}t = \int_0^{10} e^{-0.05t} \mathrm{d}t + e^{-0.5} \int_{10}^\infty e^{-0.1(t-10)} \mathrm{d}t = \frac{1}{0.05}\left(1 - e^{-0.5}\right) + \frac{e^{-0.5}}{0.10} = 20 - 10e^{-0.5}$$

IMM - DTU

02405 Probability
2003-10-14
BFN/bfn

Solution for exercise 4.4.1 in Pitman

We apply boxed results page 304. First we introduce $Y = g(X) = cX$ and note that $g()$ is strictly increasing. We have

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } 0 < x, \qquad y = g(x) = c \cdot x, \qquad x = \frac{y}{c}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = c$$

Inserting in the formula

$$f_Y(y) = \frac{\lambda e^{-\lambda \frac{y}{c}}}{c} = \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} \qquad 0 < y < 1$$

such that Y follows an exponential distribution with parameter(intensity) $\frac{\lambda}{c}$.

Alternative solution using cumulative distribution - section 4.5

We define a new random variable $Y = cX$. The distribution of Y

$$P(Y \leq y) = P(cX \leq y) = P\left(X \leq \frac{y}{c}\right) = 1 - e^{-\lambda \frac{y}{c}} = 1 - e^{-\frac{\lambda}{c} y}$$

IMM - DTU

02405 Probability
2003-10-14
BFN/bfn

Solution for exercise 4.4.2 in Pitman

The result is obvious, think of the factor λ as a change of scale, i.e. time measured in hours rather than seconds, length measured in centimeters rather than inches etc. A formal proof is as follows.

Alternative solution using cumulative distribution - section 4.5

$$P(T \leq t) = P\left(\frac{T_1}{\lambda} \leq t\right) = P(T_1 \leq \lambda t)$$

Now inserting in the expression given by (2) page 286 shows that T is Gamma (r, λ) distributed.

IMM - DTU

02405 Probability
2003-10-15
BFN/bfn

Solution for exercise 4.4.3 in Pitman

First we introduce $Y = g(U) = U^2$ and note that $g()$ is strictly increasing on $]0,1[$. We then apply the formula in the box on page 304. In our case we have

$$f_X(x) = 1 \text{ for } 0 < x < 1, \qquad y = g(x) = x^2, \qquad x = \sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 2x = 2\sqrt{y}$$

Inserting in the formula

$$f_Y(y) = \frac{1}{2\sqrt{y}} \qquad 0 < y < 1$$

Alternative solution using cumulative distribution - section 4.5

$$F_{U^2}(y) = P(U^2 \le y) = P(U \le \sqrt{y}) = \sqrt{y}$$

The last equality follows from the cumulative distribution function (CDF) of a Uniformly distributed random variable (page 487). The density is derived from the CDF by differentation (page 313) and

$$f_{U^2}(y) = \frac{\mathrm{d}F_{U^2}(y)}{\mathrm{d}y} = \frac{1}{2\sqrt{y}}, 0 < y < 1$$

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.4.6 in Pitman

We have

$$\tan(\Phi) = y$$

and use the change of variable result page 304 to get

$$\frac{\mathrm{d}\tan(\Phi)}{\mathrm{d}\Phi} = 1 + \tan(\Phi)^2 = 1 + y^2$$

Now inserting into the formula page 304 we get

$$f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2}, -\infty < y < \infty$$

The function is symmetric ($f_Y(y) = f_Y(-y)$) since $(-y)^2 = y^2$, but

$$\int_0^a y \cdot \frac{1}{\pi}\frac{1}{1+y^2}\mathrm{d}y = \frac{1}{2\pi}\ln(1+a^2) \to \infty \text{ for } a \to \infty$$

The integral $\int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y$ has to converge absolutely for $E(Y)$ to exist, i.e. $E(Y)$ exists if and only if $E(|Y|)$ exists (e.g. page 263 bottom).

Solution for exercise 4.4.9 in Pitman

Question a) Using the one-to-one change of variable results page 304 we get $Y = g(T) = T^\alpha$, $T = Y^{\frac{1}{\alpha}}$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha t^{\alpha-1}$

$$f_Y(y) = \lambda\alpha t^{\alpha-1}e^{-\lambda t^\alpha}\frac{1}{\alpha t^{\alpha-1}} = \lambda e^{-\lambda y}$$

the exponential density.

Question b) Once again using the one-to-one change of variable results page 304 we get $T = g(U) = (-\frac{1}{\lambda}\ln(U))^{\frac{1}{\alpha}}, U = e^{-\lambda T^\alpha}$, $\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| = \frac{1}{\lambda\alpha}\frac{1}{u}(-\frac{1}{\lambda}\ln(U))^{\frac{1}{\alpha}-1}$

$$f_T(t) = 1\frac{1}{\frac{1}{\lambda\alpha}\frac{1}{u}(-\ln(u))^{\frac{1}{\alpha}-1}} = \lambda\alpha e^{-t^\alpha}(t^\alpha)^{1-\frac{1}{\alpha}} = \lambda\alpha t^{\alpha-1}e^{-t^\alpha}$$

a Weibull(λ, α) density.

Alternative solution using cumulative distribution - section 4.5

Question a)

$$P(T^\alpha \leq t) = P(T \leq t^{\frac{1}{\alpha}})$$

Since T has the Weibull distribution we find

$$P(T \leq x) = F_{Wei}(x) = \int_0^x \lambda\alpha u^{\alpha-1}e^{-\lambda u^\alpha}\mathrm{d}u = \left[e^{-\lambda u^\alpha}\right]_{u=0}^{u=x} = 1 - e^{-\lambda x^\alpha}$$

Now inserting $x = t^{\frac{1}{\alpha}}$ we get

$$P(T^\alpha \leq t) = 1 - e^{-\lambda\left(t^{\frac{1}{\alpha}}\right)^\alpha} = 1 - e^{-\lambda t}$$

which shows us that T^α has an exponential distribution.

Question b) We examine the random variable $Y = (-\lambda^{-1}\ln(U))^{\frac{1}{\alpha}}$.

$$P(Y \leq y) = P((-\lambda^{-1}\ln(U))^{\frac{1}{\alpha}} \leq y) = P((-\lambda^{-1}\ln(U)) \leq y^\alpha)$$

$$= P((\ln(U)) \geq -\lambda y^\alpha) = P(U \geq e^{-\lambda y^\alpha}) = P(1 - U \leq 1 - e^{-\lambda y^\alpha})$$

Now since U is uniformly distributed so is $1-U$ and we deduce

$$P(Y \le y) = P(U \le 1 - e^{-\lambda y^{\alpha}}) = 1 - e^{-\lambda y^{\alpha}}$$

where the last equality follows from page 487 (cumulative distribution function), which was to be shown.

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.4.10 in Pitman

Question a) First we introduce $Y = g(Z) = |Z|$ and note that $g()$ is strictly increasing on $]0, \infty[$, and strictly decreasing on $]-\infty, 0[$. We then apply the formula in the box on page 304 and the many to one result on the top of page 307. In our case we have

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad y = g(z) = |z|, \qquad \left|\frac{\mathrm{d}y}{\mathrm{d}z}\right| = 1$$

Inserting in the formula

$$f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}y^2} \qquad 0 < y < \infty$$

Question b) We introduce $Y = g(Z) = Z^2$.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad y = g(z) = z^2, \qquad z = \sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 2z = 2\sqrt{y}$$

Inserting in the boxed formula page 304 and once again using the many to one extension.

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \qquad 0 < y < \infty$$

This is a special case of the χ^2 distribution, here with 1 degree of freedom. The general case is introduced page 365. The distribution is extremely important in statistics (and probability).

Question c) We introduce $Y = g(Z) = \frac{1}{Z}$. With $Z = \frac{1}{Y}$ and $\frac{\mathrm{d}g(z)}{\mathrm{d}z} = -\frac{1}{z^2}$ we get

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{1}{\frac{1}{z^2}} = \frac{1}{y^2\sqrt{2\pi}} e^{-\frac{1}{2y^2}}, -\infty < y < 0; 0 < y < \infty$$

Question d) We introduce $Y = g(Z) = \frac{1}{Z^2}$. With $Z = \frac{1}{\sqrt{Y}}$ and $\frac{\mathrm{d}g(z)}{\mathrm{d}z} = -2\frac{1}{z^3}$ we get

$$f_Y(y) = 2\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{1}{\frac{2}{z^3}} = \frac{1}{y\sqrt{y}\sqrt{2\pi}} e^{-\frac{1}{2y}}, 0 < y < \infty$$

the factor 2 stems from the many to one situation page 306/307.

IMM - DTU

02405 Probability
2004-23-3
BFN/bfn

Solution for exercise 4.5.1 in Pitman

Question a) The survival function of the exponenital distibution is - page 279 -

$$P(T > t) = e^{-\lambda t}$$

thus the cumulative distibution function $F(t)$ is

$$F(t) = P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t}$$

IMM - DTU

02405 Probability
2004-23-3
BFN/bfn

Solution for exercise 4.5.2 in Pitman

Question a)

$$P(X \leq x) = \sum_{i=0}^{x} \binom{3}{i} \frac{1}{8}$$

Question b)

$$P(X \leq x) = 1 - P(X > x) = 1 - P(X \geq x+1) = 1 - \frac{1}{2^x}$$

IMM - DTU

02405 Probability
2003-10-16
BFN/bfn

Solution for exercise 4.5.4 in Pitman

The operations considered are shifting (addition of b) and scaling (multiplication by a). We introduce $Y = aX + b$. The distribution $F_Y(y)$ of Y is given by

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(aX \leq y - b)$$

For $a > 0$ we get

$$F_Y(y) = P\left(X \leq \frac{y-b}{a}\right) = F\left(\frac{y-b}{a}\right)$$

For $a < 0$ we get

$$F_Y(y) = P\left(X \geq \frac{y-b}{a}\right) = 1 - P\left(X \leq \frac{y-b}{a}\right) = 1 - F\left(\frac{y-b}{a}\right)$$

Solution for exercise 4.5.6 in Pitman

Question a) From the definition of the cumulative distribution function page 313 we get

$$P\left(X \geq \frac{1}{2}\right) = 1 - P\left(X < \frac{1}{2}\right) = 1 - P\left(X \leq \frac{1}{2}\right)$$

where the last equality is true for continuous distributions.

$$P\left(X \geq \frac{1}{2}\right) = 1 - P\left(X \leq \frac{1}{2}\right) = 1 - F\left(\frac{1}{2}\right) = \frac{7}{8}$$

Question b) The density is the first derivative of the CDF for a continuous distribution (page 313), thus

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} = 3x^2$$

Question c) We calculate the mean from the definition page 261

$$E(X) = \int_0^1 x f(x) \mathrm{d}x = \int_0^1 x \cdot 3x^2 \mathrm{d}x = \frac{3}{4}$$

Question d) The variables Y_1, Y_2, and Y_3 are all uniformly distributed with CDF $F_Y(y) = y$ (see eg. page 315). The discussion on the distribution of maximum of n indpendent random variables page 316 tells us that $Z = \max(Y_1, Y_2, Y_3)$ with CDF $F_Z(z)$

$$F_Z(z) = P(Z \leq z) = P(\max(Y_1, Y_2, Y_3) \leq z) = P(Y_1 \leq z, Y_2 \leq z, Y_3 \leq z)$$
$$= P(Y_1 \leq z)P(Y_2 \leq z)P(Y_3 \leq z) = z^3$$

IMM - DTU

02405 Probability
2003-10-29
BFN/bfn

Solution for exercise 4.5.7 in Pitman

Question a) The exercise is closely related to exercise 4.4.9 page 310, as it is the inverse problem in a special case. We apply the standard change of variable method page 304

$$Y = \sqrt{T}, T = Y^2, \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\sqrt{t}}$$

$$f_Y(y) = 2\lambda \cdot y e^{-\lambda y^2}$$

a Weibull distribution. See e.g. exercise 4.3.4 page 301 and exercise 4.4.9 page 310.

Question b)

$$\int_0^\infty 2\lambda y^2 e^{-\lambda y^2} \mathrm{d}y = \int_{-\infty}^\infty \lambda y^2 e^{-\lambda y^2} \mathrm{d}y$$

We note the similarity with the variance of an unbiased (zero mean) normal variable.

$$\int_{-\infty}^\infty \lambda y^2 e^{-\lambda y^2} \mathrm{d}y = \lambda \int_{-\infty}^\infty y^2 \sqrt{\frac{2\pi}{2\pi}} \sqrt{\frac{\frac{1}{2\lambda}}{\frac{1}{2\lambda}}} e^{-\frac{1}{2}\frac{y^2}{\frac{1}{2\lambda}}} \mathrm{d}y = \lambda \sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^\infty y^2 \frac{1}{\sqrt{2\pi}} \frac{1}{\frac{1}{\sqrt{2\lambda}}} e^{-\frac{1}{2}\frac{y^2}{\frac{1}{2\lambda}}} \mathrm{d}y$$

the integral is the expected value of Z^2, where Z is *normal* $\left(0, \frac{1}{2\lambda}\right)$ distributed. Thus the value of the integral is $\frac{1}{2\lambda}$ Finally we get

$$E(Y) = \sqrt{\lambda\pi} E(Z^2) = \sqrt{\lambda\pi} Var(Z)$$

$$= \sqrt{\lambda\pi} \frac{1}{2\lambda} == \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = 0.51 \qquad \text{with } \lambda = 3$$

Question c) We apply the inverse distribution function method suggested page 320-323. Thus

$$U = 1 - e^{-\lambda X} \Rightarrow X = -\frac{1}{\lambda} \ln(1-U)$$

Now $1-U$ and U are identically distributed such that we can generate an exponential X with $X = -\frac{1}{\lambda}\ln(U)$. To generate a Weibull $(\alpha = 2)$ distributed Y we take the square root of X, thus $Y = \sqrt{-\frac{1}{\lambda}\ln(1-U)}$.

IMM - DTU

02405 Probability
2003-10-17
BFN/bfn

Solution for exercise 4.5.8 in Pitman

We let X_i denote the lifetime of the i'th component, and S denote the lifetime of the system.

Question a) We have the maximum of two exponential random variables $S = \max(X_1, X_2)$.

$$P(S \leq t) = P(\max(X_1, X_2) \leq t) = \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right)$$

from page 316 and example 4 page 317/318. Thus

$$P(S > t) = 1 - \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

Question b) In this case we have $S = \min(X_1, X_2)$ and we apply the result for the minimum of random variables page 317. The special case of two exponentials is treated in example 3 page 317

$$P(S \leq t) = 1 - e^{-(\lambda_1+\lambda_2)t}$$

Question c) From the system design we deduce $S = \max(\min(X_1, X_2), \min(X_3, X_4))$ such that

$$P(S \leq t) = \left(1 - e^{-(\lambda_1+\lambda_2)t}\right)\left(1 - e^{-(\lambda_3+\lambda_4)t}\right)$$

Question d) Here $S = \min(\max(X_1, X_2), X_3)$ such that

$$P(S \leq t) = 1 - \left(1 - \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right)\right)e^{-\lambda_3 t} = 1 - e^{-(\lambda_1+\lambda_3)t} - e^{-(\lambda_2+\lambda_3)t} + e^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

IMM - DTU

02405 Probability
2006-11-3
BFN/bfn

Solution for exercise 4.6.1 in Pitman

We introduce the random variables $X_i; i = 1, 2, 3, 4$ for the arrival time of the i'th person. For convenience X_i will be the deviation from 12 noon measured in minutes.

Question a) Since X_i are continuous random variables the question can be stated as

$$P(\min_i X_i < -10) = P(\min_i X_i \leq -10)$$

From the result page 317 and the normality of the X_i's we get

$$P(\min_i X_i \leq 10) = 1 - \left(1 - \Phi\left(\frac{10}{5}\right)\right)^4 = 1 - 0.9772^4 = 0.088$$

(compared with the probability 0.0228 that a specific person will arrive before 11.50)

Question b) This question can be stated as

$$P(\max_i (X_i) > 15) = 1 - P(\max_i (X_i) \leq 15) = 1 - \Phi\left(\frac{-15}{5}\right)^4 = 1 - 0.9987^4 = 0.0052$$

from the result regarding the distribution of the maximum of indpendent random variables page 316.

Question c) The question regards the second order distribution i.e. the distribution of $X_{(2)}$ where $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq X_{(4)}$. The expression for this density is stated page 326. With $x = 0$, $\mathrm{d}x = 2 \cdot \frac{1}{6}$, and $f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{5}\right)^2}$ (page 267) we get

$$P\left(-\frac{1}{6} \leq X_{(2)} \leq \frac{1}{6}\right) \doteq f_{(2)}(0)\frac{2}{6} = 4\begin{pmatrix} 3 \\ 1 \end{pmatrix} \frac{1}{5\sqrt{2\pi}} \frac{2}{6} \frac{1}{2} \left(\frac{1}{2}\right)^2 = 0.0399$$

(we have used $F(0) = \Phi\left(\frac{0-0}{5}\right) = \frac{1}{2}$)

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 4.6.3 in Pitman

Question a)

$$P(U_{(1)} \geq x, U_{(n)} \leq y) = P(x \leq U_1 \leq y, x \leq U_2 \leq y, \ldots x \leq U_n \leq y) = (y-x)^n$$

Question b)

$$P(U_{(1)} \geq x, U_{(n)} > y) = P(U_{(1)} \geq x) - P(U_{(1)} \geq x, U_{(n)} \leq y) = (1-x)^n - (y-x)^n$$

Question c)

$$P(U_{(1)} \leq x, U_{(n)} \leq y) = P(U_{(n)} \leq y) - P(U_{(1)} \geq x, U_{(n)} \leq y) = y^n - (y-x)^n$$

Question d)

$$1 - (1-x)^n - y^n + (y-x)^n$$

Question e)

$$\binom{n}{k} x^k (1-y)^{n-k}$$

Question f)

$$k < x, n-k-1 > y$$

one in between

IMM - DTU

02405 Probability
2005-4-7
BFN/bfn

Solution for review exercise 1 (chapter 1) in Pitman

Solution for exercise 4.6.4 in Pitman

Question a)

$$P(Z=1)=P(S<T)=P(S\le T)=P(T\le S)=P(Z=0)\Rightarrow P(Z=1)=P(Z=0)=\frac{1}{2}$$

Question b) It is intuitevly tempting to claim that X and Z are independent. This is an example where intuition is correct. However one should be careful and should be able to verify with rigourous arguments.

$$P(X\le x|Z=1)=\frac{P(X\le x,Z=1)}{P(Z=1)}=\frac{P(S\le x,S<T)}{\frac{1}{2}}$$

Now

$$P(X\le x)=P(S\le x,S<T)+P(T\le x,T<S)=2P(T\le x,T<S)$$

Inserting we get

$$P(X\le x|Z=1)=\frac{\frac{P(X\le x)}{2}}{\frac{1}{2}}=P(X\le x)$$

A similar argument shows the independence of Y and Z.

Question c) Indpendence between which variable attains the k'th order statistic and the value of the k'th order statistic.

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 4.6.5 in Pitman

Question a) The probability $P(X_i \leq x) = x$ since X_i is uniformly distributed. The number N_x of X_i's less than or equal to x follows a binomial distribution $bin(n, x)$ since the X_i are independent. The event $\{X_{(k)} \leq x\}$ corresponds to $\{N_x \geq k\}$. We get

$$P(X_{(k)} \leq x) = P(N_x \geq k) = \sum_{i=k}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

Question b) From the boxed result at the bottom of page 327 we have that ($X_{(k)}$ has $beta(k, n-k+1)$ distribution. Substituting $r = k$ and $s = n-k+1$ we get

$$P(X_{(k)} \leq x) = \sum_{i=r}^{r+s-1} \binom{r+s-1}{i} x^i (1-x)^{s+r-i-1}$$

which is the stated result.

Question c) The $beta(r, s)$ density is

$$f(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} = \frac{1}{B(r,s)} x^{r-1} \sum_{i=0}^{s-1} \binom{s-1}{i} (-x)^i$$

Now

$$P(X_{(k)} \leq x) = \int_0^x f(x)\mathrm{d}x = \int_0^x \frac{1}{B(r,s)} u^{r-1} \sum_{i=0}^{s-1} \binom{s-1}{i} (-u)^i \mathrm{d}u$$

$$= \frac{1}{B(r,s)} \sum_{i=0}^{s-1} \int_0^x \binom{s-1}{i} (-u)^{r+i-1} \mathrm{d}u = \frac{x^r}{B(r,s)} \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{(-x)^i}{r+i}$$

as was to be proved.

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 5.1.1 in Pitman

Question a) Consider the area of the support for the density to get

$$P(X > 1) = \frac{\frac{5}{2}}{6} = \frac{5}{12} \qquad P(X \leq 1) = \frac{7}{12}$$

or integration of

$$\int_0^1 \int_x^4 \frac{1}{6} \mathrm{d}y \mathrm{d}x = \int_0^1 \frac{4-x}{6} \mathrm{d}x = \frac{4 - \frac{1}{2}}{6} = \frac{7}{12}$$

Question b)

$$\int_1^2 \int_x^{x^2} \frac{1}{6} \mathrm{d}y \mathrm{d}x = \frac{1}{6} \int_1^2 (x^2 - x) \mathrm{d}x = \frac{1}{6} \left[\frac{1}{3} x^3 - \frac{1}{2} x^2 \right]_{x=1}^{x=2} = \frac{5}{36}$$

(note that $x^2 < x$ for $0 < x < 1$)

IMM - DTU

02405 Probability
2003-11-20
BFN/bfn

Solution for exercise 5.1.2 in Pitman

Question a) Let X_i denote the random value of the result of the i'th measurement $(i = 1, 2)$. The density of X_i is given by

$$f(x) = \begin{cases} 5 & l - \frac{1}{10} < x < l + \frac{1}{10} \\ 0 & \text{elsewhere} \end{cases}$$

with cumulative distribution function

$$F(x) = \begin{cases} 0 & x \leq l - \frac{1}{10} \\ 5\left(x - \left(l - \frac{1}{10}\right)\right) & l - \frac{1}{10} < x < l + \frac{1}{10} \\ 1 & l + \frac{1}{10} \leq x \end{cases}$$

We find directly or using

$$P\left(l - \frac{1}{100} < X_i < l + \frac{1}{100}\right) = F\left(l + \frac{1}{100}\right) - F\left(l - \frac{1}{100}\right) = 0.1$$

Question b) This is example 3 page 343 with different parameters.

$$1 - 25\left(\frac{19}{100}\right)^2 = 1 - 0.9025 = 0.0975$$

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 5.1.4 in Pitman

Question a) This is Example 3 page 343 with different numbers

$$P(|Y-X| \leq 0.25) = 1 - 2\frac{1}{2}\left(\frac{3}{4}\right)^2 = \frac{7}{16}$$

Question b) We see that the probability can be rewritten This is example 2 page 343 with different values. We get

$$1 - \frac{1}{2}\frac{3}{4} - \frac{1}{2}\frac{4}{5} = \frac{9}{40}$$

Question c)

$$P(Y \geq X | Y > 0.25) = \frac{P(Y \geq X, Y > 0.25)}{P(Y > 0.25)} = \frac{P(Y \geq X) - P(Y \geq X, Y \leq 0.25)}{P(Y > 0.25)}$$

$$= \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{4}\right)^2}{\frac{3}{4}} = \frac{5}{8}$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 5.1.5 in Pitman

We note that the percentile U of a randomly chosen student is uniformly$(0,1)$ distributed.

Question a)

$$P(U > 0.9) = 1 - P(U \leq 0.9) = 0.9$$

Question b) The question is Example 3 page 343 the probability of a meeting with different parameters. Denoting U_1 and U_2 respectively as the rank of the two students

$$P(|U_1 - U_2| > 0.1) = 0.9^2 = 0.81$$

IMM - DTU

02405 Probability
2003-11-2
BFN/bfn

Solution for exercise 5.1.6 in Pitman

Question a) Define A_1 and A_2 as the arrival times of Jack and Jill respectively. The probability in question is

$$P(A_2 > A_1 + 2) = \frac{1}{2}\left(\frac{13}{15}\right)^2 = \frac{169}{450} = 0.3756$$

Question b) As in the textbook we will denote $A_{(1)}$ as the smallest and $A_{(10)}$ as the largest of the ten arrival times. The probability in question is $P(A_{(1)} < 5, 10 < A_{(10)})$. This probability has been analyzed in exercise 4.6.3 d) page 330. From that exercise we derive

$$P(A_{(1)} < 5, 10 < A_{(10)}) = 1-(1-x)^n-y^n+(y-x)^n = 1-\left(\frac{2}{3}\right)^{10}-\left(\frac{2}{3}\right)^{10}+\left(\frac{1}{3}\right)^{10}$$

$$= 1 - \frac{1}{3^{10}}(2^{11} - 1)$$

can be solved this way using E.4.6.3, I am looking for a shortcut before finishing the solution. However 4.6.3 is scheduled two weeks before this one.

Solution for exercise 5.2.1 in Pitman

Question a) A nice drawing The area of the figure (shaded area) is 1.

$$f(x, y) = 1, \qquad 0 < |y| < x$$

Question b) We find the marginal distribution of X by integrating over y for fixed x (page 349)

$$f_X(x) = \int_{-x}^{x} 1 \cdot \mathrm{d}y = 2x, \qquad 0 < x < 1$$

Similarly for positive y

$$f_Y(y) = \int_{y}^{1} 1 \cdot \mathrm{d}x = 1 - y, \qquad 0 < y < 1$$

and for negative y

$$f_Y(y) = \int_{-y}^{1} 1 \cdot \mathrm{d}x = 1 + y, \qquad -1 < y < 0$$

leading to a general expression for $f_Y(y)$

$$f_Y(y) = 1 - |y|, \qquad -1 < y < 1$$

Question c) No. (e.g. $P\left(|Y| > \frac{1}{2} | X < \frac{1}{2}\right) = 0 \neq \frac{1}{4} = P\left(|Y| > \frac{1}{2}\right)$.

Question d) From the definiton of $E(X)$ page 261 (page 332)

$$E(x) = \int_{0}^{1} x \cdot 2x \mathrm{d}x = \frac{2}{3}$$

The distribution of Y is symmetric around 0 so $E(Y) = 0$.

IMM - DTU

02405 Probability
2003-10-16
BFN/bfn

Solution for exercise 5.2.4 in Pitman

We can rewrite the density

$$f(x,y) = 2e^{-2x}3e^{-3y}$$

to see that X and Y are independent expontially distributed random variables which basically solves a)-c). Alternatively:

Question a) The area B page 349 is defined by the rectangle $0 < u < x, 0 < v < y$.

$$P(X \leq x, Y \leq y) = \int_0^x \int_0^y f(u,v)\mathrm{d}v\mathrm{d}u = \int_0^x \int_0^y 2e^{-2u}3e^{-3v}\mathrm{d}v\mathrm{d}u$$

$$= \int_0^x 2e^{-2u}\left(1 - e^{-3y}\right)\mathrm{d}u = \left(1 - e^{-2x}\right)\left(1 - e^{-3y}\right)$$

Question b)

$$f_X(x) = 2e^{-2x}$$

Question c)

$$f_Y(y) = 3e^{-3y}$$

Question d) The variables X and Y are independent since

$$f(x,y) = f_X(x)f_Y(y)$$

for all (x,y).

IMM - DTU

02405 Probability
2005-10-29
BFN/bfn

Solution for exercise 5.2.6 in Pitman

Question a) Draw a small figure showing the area of integration. Using page 349 we get

$$P(Y > 2X) = \int_0^{\frac{1}{2}} \int_{2x}^{1} 90(y-x)^8 \mathrm{d}y\mathrm{d}x = \int_0^{\frac{1}{2}} \left[10(y-x)^9\right]_{y=2x}^{y=1} \mathrm{d}x = \int_0^{\frac{1}{2}} (10(1-x)^9 - 10x^9)\mathrm{d}x$$

$$= \left[-(x-1)^{10} - x^{10}\right]_{x=0}^{x=\frac{1}{2}} = 1 - 2\left(\frac{1}{2}\right)^{10}$$

Question b) The marginal density of X is given by (using page 349)

$$f_X(x) = \int_x^1 90(y-x)^8 \mathrm{d}y = 10(1-x)^9$$

with CDF

$$F_X(x) = \int_0^x 10(1-u)^9 \mathrm{d}u = 1 - (1-x)^{10}$$

The marginal density of Y is given by (using page 349)

$$f_Y(y) = \int_0^y 90(y-x)^8 \mathrm{d}y = 10y^9$$

with CDF

$$F_Y(y) = \int_0^y 10u^9 \mathrm{d}u = y^{10}$$

Question c) Maximum and minimum (see exercise 4.6.3). Also note that the marginal distributions are those of max and min from page 316/317.

IMM - DTU

02405 Probability
2003-10-17
BFN/bfn

Solution for exercise 5.2.7 in Pitman

We denote the radius of the circle by ρ. The are of the circle is $\pi\rho^2$. If a chosen point is within radius r it has to be within the circle of radius r with area πr^2. We find the probability as the fraction of these two areas

$$F_R(r) = P(R_1 \leq r) = \frac{r^2}{\rho^2}$$

with density (page 333)

$$f_R(r) = \frac{\mathrm{d}F_R(r)}{\mathrm{d}r} = \frac{2r}{\rho^2}$$

With R_1 and R_2 indpendent we have the joint density from (2) page 350

$$f(r_1, r_2) = \frac{4r_1r_2}{\rho^4}$$

We now integrate over the set $r_2 < \frac{r_1}{2}$ (page 349) to get

$$P\left(R_2 \leq \frac{R_1}{2}\right) = \int_0^{\rho}\int_0^{\frac{r_1}{2}} \frac{4r_1r_2}{\rho^4}\mathrm{d}r_2\mathrm{d}r_1 = \frac{1}{2\rho^4}\int_0^{\rho} r_1^3\mathrm{d}r_1 = \frac{1}{8}$$

IMM - DTU

02405 Probability
2003-11-20
BFN/bfn

Solution for exercise 5.2.8 in Pitman

Question a) We find the marginal density of Y by integrating over x (page 349)

$$f_Y(y) = \int_{-y}^{y} c(y^2 - x^2)e^{-y}\mathrm{d}x = c\frac{4}{3}y^3 e^{-y}$$

We recognize this as a gamma density (1) page 286 with $\lambda = 1$ and $r = 4$ thus $c = \frac{1}{8}$

Question b) With $Z = g(Y) = 4Y^3$, $\frac{\mathrm{d}g(y)}{\mathrm{d}y} = 12y^2$, $Y = \left(\frac{Z}{4}\right)^{\frac{1}{3}}$, using the boxed result page 304 we get

$$f_Z(z) = \frac{y^3}{6}e^{-y}\frac{1}{12y^2} = \frac{\left(\frac{z}{4}\right)^{\frac{1}{3}}}{72}e^{-\left(\frac{z}{4}\right)^{\frac{1}{3}}}$$

Question c) We have $|X| \leq |Y| = Y$. Thus $E(|X|) \leq E(Y) = 4$.

IMM - DTU

02405 Probability
2003-11-22
BFN/bfn

Solution for exercise 5.2.11 in Pitman

Question a)

$$E(X+Y) = E(X) + E(Y) = 1.5$$

from the general rule of the expectation of a sum.

Question b)

$$E(XY) = E(X)E(Y) = 0.5$$

by the independe of X and Y.

Question c)

$$E((X-Y)^2) = E(X^2 + Y^2 - 2XY) = E(X^2) + E(Y^2) - 2E(XY)$$

$$= (Var(X) + (E(X))^2) + (Var(Y) + (E(Y))^2) - 2E(XY) = \frac{1}{12} + \frac{1}{4} + 1 + 1 - 1 = \frac{4}{3}$$

from the general rule of the expectation of a sum, the computational formula for the variance, and the specific values for the uniform and exponential distributions.

Question d)

$$E\left(X^2 e^{2Y}\right) = E(X^2)E\left(e^{2Y}\right)$$

We recall the general formula for $E(g(Y))$ from page 263 or 332

$$E(g(Y)) = \int_y g(y) f(y) \mathrm{d}y$$

where $f(y)$ is the density of Y. Here Y is $exponential(1)$ distributed with density $f(y) = 1 \cdot e^{-1 \cdot y}$. We get

$$E\left(e^{2Y}\right) = \int_0^\infty e^{2y} 1 \cdot e^{-y} \mathrm{d}y = \infty$$

thus $E\left(X^2 e^{2Y}\right)$ is undefined (∞).

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 5.2.15 in Pitman

Question a)

$$P(a < X \leq b, c < Y \leq d) = P(X \leq b, c < Y \leq d) - P(X \leq a, c < Y \leq d)$$

$$= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - (P(X \leq a, Y \leq d) - P(X \leq a, Y \leq c))$$
$$= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - P(X \leq a, Y \leq d) + P(X \leq a, Y \leq c)$$
$$= F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

This relation can also be derived from geometric considerations.

Question b)

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\mathrm{d}u\mathrm{d}v$$

Question c)

$$f(x,y) = \frac{d^2F(x,y)}{\mathrm{d}x\mathrm{d}y}$$

from the fundamental theorem of calculus.

Question d) The result follows from (2) page 350 by integration.

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(x)f_Y(y)\mathrm{d}y\mathrm{d}x = \int_{-\infty}^{x} f_X(x)\mathrm{d}x \int_{-\infty}^{y} f_Y(y)\mathrm{d}y = F_X(x)F_Y(y)$$

Alternatively define the indicator $I(x,y)$ variables such that $I(x,y) = 1$ if $X \leq x$ and $Y \leq y$ and 0 otherwise. Note that $F(x,y) = P(I(x,y) = 1) = E(I(x,y))$ and apply the last formula on page 349.

Question e) See also exercise 4.6.3 c). We find

$$F(x,y) = P(U_{(1)} \leq x, U_{(n)} \leq y) = P(U_{(n)} \leq y) - P(U_{(1)} > x, U_{(n)} \leq y)$$

$$P(U_{(n)} \leq y) - P(x < U_1 \leq y, x < U_2 \leq y, \ldots, x < U_n \leq y) = y^n - (y-x)^n$$

We find the density as

$$\frac{d^2F(x,y)}{\mathrm{d}x\mathrm{d}y} = n(n-1)(y-x)^{n-2}$$

Solution for exercise 5.3.1 in Pitman

Question a) With R being the distance from the bulls eye to the shot we have (page.360 line b.4)

$$P(R \leq r) = F_R(r) = 1 - e^{-\frac{1}{2}r^2}$$

thus

$$F_R\left(\frac{1}{2}\right) = 1 - e^{-\frac{1}{8}} = 0.1175$$

Question b)

$$P(1 \leq R \leq 2) = F_R(2) - F_R(1) = e^{-\frac{1}{2}} - e^{-2} = 0.4712$$

we get $\frac{1}{4} \cdot 0.4712 = 0.1178$ due to the symmetry.

Question c) We are considering only the second coordinate, which is a standard normal variable. The mean of the absolute value is given page 484.

$$\sqrt{\frac{2}{\pi}}$$

Question d) This is the probability that the absolute value of the first coordinate is less than or equal to $r = 1.1777\ldots$.

$$(2\Phi(r) - 1) \simeq (2\Phi(1.18) - 1) = 0.762$$

Question e) This is the probability that the absolute value of largest of the two coordinates are less than or equal to r.

$$(2\Phi(r) - 1)^2 \simeq (2\Phi(1.18) - 1)^2 = 0.581$$

Question f) Use rotational symmetry and find similarly to e)

$$(2\Phi\left(\frac{r}{\sqrt{2}}\right) - 1)^2 = 0.352$$

Question g)

$$\frac{1}{2}(2\Phi(r) - 1)^2 = 0.29$$

IMM - DTU

02405 Probability
2004-4-11
BFN/bfn

Solution for exercise 5.3.3 in Pitman

Question a)

$$P(W+X>Y+Z+1) = P(W+X-Y-Z>1)$$

The variable $V = (W+X-Y-Z) \in Normal(0,4)$ Thus

$$P(W+X>Y+Z+1) = P(V>1) = P\left(\frac{V-0}{2} > \frac{1}{2}\right) = 1-\Phi\left(\frac{1}{2}\right) = 1-0.6915$$

Question b)

$$P(4X+3Y<Z+W) = P(4X+3Y-Z-W<0) = 0.5$$

Question c)

$$E(4X+3Y-2Z^2-W^2+8) = 4E(X)+3E(Y)-2E(Z^2)-E(W^2)+8$$

from the standard result: the expectation of a linear expression is the linear expression of the expectations.

$$4E(X)+3E(Y)-2E(Z^2)-E(W^2)+8 = -2-1+8 = 5$$

since X, Y, Z, W are standard normal variables.

Question d)

$$Var(3Z-2X+Y+15) = 9Var(Z)+4Var(X)+Var(Y) = 14$$

since X, Y, Z are independent and standard normal variables.

$$SD(3Z-2X+Y+15) = \sqrt{Var(3Z-2X+Y+15)} = \sqrt{14}$$

IMM - DTU

02405 Probability
2004-4-15
BFN/bfn

Solution for exercise 5.3.6 in Pitman

Question a)

$$P(N(0,13) > 5) = 1 - \Phi\left(\frac{5}{\sqrt{13}}\right)$$

Question b)

$$1 - (1 - \Phi(1))^2$$

Question c) Drawing helpful, suggests that the following should be true

$$\Phi(1) - \Phi(-1)$$

Question d)

$$P(1 > \max(X,Y) - \min(X,Y) = P(1 > |X - Y|) = \Phi\left(\frac{1}{\sqrt{2}}\right) - \Phi\left(\frac{-1}{\sqrt{2}}\right)$$

IMM - DTU

02405 Probability
2003-10-17
BFN/bfn

Solution for exercise 5.3.9 in Pitman

Question a) We will do all the calculations in inches, noting that 1 foot is 12 inches. We have the distribution of the largest observation from page 316

$$P(X_{(100)} > 76) = 1-P(X_{(100)} \leq 76) = 1-\left(\Phi\left(\frac{76-70}{2}\right)\right)^{100} = 1-0.9987^{100} = 0.122$$

(Pitman 0.1307??? which is what you would get with $P(X_{(100)} \leq 76) = 0.9986$)

Question b) The average of $\bar{X}$ of the 100 observations is given by $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$. We have from the boxed result page 363 that $\sum_{i=1}^{100} X_i$ is normally distributed implying that $\bar{X}$ is normally distributed. We find mean and standard deviation of $\bar{X}$ using the Square Root Law page 194 such that

$$P(\bar{X} > 70.5) = 1-\Phi\left(\frac{70.5-70}{\frac{2}{\sqrt{100}}}\right) = 1-\Phi(2.5) = 0.0062$$

Question c) Page the Central Limit Theorem (e.g. page 386) can be applied in case b). Limit theorems exists for maximum and minimum of random variables (extreme value distributions). These results depend on the specific form of the distribution of the individual X_i's. One can easily construct counterexamples to disprove the generality of a), like uniformly distributed X_i's.

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 5.3.12 in Pitman

Question a) Let the coordinates shot i be denoted by (X_i, Y_i). The difference between two shots $(X_2 - X_1, Y_2 - Y_1)$ is two independent normally distributed random variables with mean 0 and variance 2. By a simple a scaling in example 1 problem 2 page 361 we get $E(D) = \sqrt{2}\sqrt{\frac{\pi}{2}} = \sqrt{\pi}$.

Question b) We have $E(D^2) = 4$ thus $Var(D) = 4 - \pi$.

IMM - DTU

02405 Probability
2003-11-22
BFN/bfn

Solution for exercise 5.3.15 in Pitman

Question a) This is exercise 4.4.10 b). We recall the result Introducing $Y = g(Z) = Z^2$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad y = g(z) = z^2, \qquad z = \sqrt{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 2z = 2\sqrt{y}$$

Inserting in the boxed formula page 304 and use the many to one extension.

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \qquad 0 < y < \infty$$

We recognize the gamma density with scale parameter $\lambda = \frac{1}{2}$ and shape parameter $r = \frac{1}{2}$ from the distribution summary page 481. By a slight reformulation we have

$$f_Y(y) = \frac{1}{2} \frac{\left(\frac{y}{2}\right)^{\frac{1}{2}-1}}{\sqrt{\pi}} e^{-\frac{y}{2}}$$

and we deduce have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Question b) The formula is valid for $n = 1$. Assuming the formula valid for odd n we get

$$Gamma\left(\frac{n+2}{2}\right) = \Gamma\left(\frac{n}{2} + 1\right)$$

The recursive formula for the gamma-function page 191 tells us that $\Gamma(r+1) = r\Gamma(r)$ and we derive

$$Gamma\left(\frac{n+2}{2}\right) = \frac{n}{2} \frac{\sqrt{\pi}(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}$$

$$\Gamma\left(\frac{n}{2}\right) = \prod_{i=1}^{\frac{n-1}{2}} \left(i - \frac{1}{2}\right)\sqrt{\pi}$$

Question c) Obvious by a simple change of variable.

Question d) From the additivity of the gamma distribution, which we can prove directly

Question e) From the interpretation as sums of squared normal variables.

Question f) The mean of a gamma (r, λ) distribution is $\frac{r}{\lambda}$, thus χ^n has mean $\frac{\frac{n}{2}}{\frac{1}{2}} = n$. The variance of a gamma (r, λ) distribution is $\frac{r}{\lambda^2}$, thus the variance of χ^n is $\frac{\frac{n}{2}}{\frac{1}{4}} = 2n$. Skewness bla bla bla

Solution for exercise 5.4.1 in Pitman

Question a) The joint density of (X_1, X_2) is

$$f(x_1, x_2) = \begin{cases} \frac{1}{2} & 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

We find

$$P(X_1 + X_2 \leq 2) = \int_{x_1 + x_2 \leq 2} \frac{1}{2} \mathrm{d}x_1 \mathrm{d}x_2$$

or use straightforward area considerations, to get $P(X_1 + X_2 \leq 2) = \frac{3}{4}$

Question b) We use the boxed expression page 372 to get

$$f_{x_1+x_2}(z) = \begin{cases} \int_0^z 1 \cdot \frac{1}{2} \mathrm{d}x_1 & 0 \leq z \leq 1 \\ \int_0^1 1 \cdot \frac{1}{2} \mathrm{d}x_1 & 1 \leq z \leq 2 \\ \int_{z-2}^1 1 \cdot \frac{1}{2} \mathrm{d}x_1 & 2 \leq z \leq 3 \end{cases}$$

thus

$$f_{x_1+x_2}(z) = \begin{cases} \frac{z}{2} & 0 \leq z \leq 1 \\ \frac{1}{2} & 1 \leq z \leq 2 \\ \frac{3-z}{2} & 2 \leq z \leq 3 \end{cases}$$

Question c)

$$F_{x_1+x_2}(z) = \int_0^z f_{x_1+x_2}(u) \mathrm{d}u = \begin{cases} \frac{z^2}{4} & 0 \leq z \leq 1 \\ \frac{2z-1}{4} & 1 \leq z \leq 2 \\ \frac{6z-z^2-5}{4} & 2 \leq z \leq 3 \end{cases}$$

Solution for exercise 5.4.2 in Pitman

Question a) Consider the joint distribution on the unit square. The area of the triangle $x+y>1, x<1, y<1$ is $\frac{1}{8}$, thus $F_{S_2}(1.5) = 1 - \frac{1}{8} = \frac{7}{8}$. Alternatively one could use the boxed result page 372 with $S_2 = X_1 + X_2$, X_i uniforum. We find

$$f_{S_2}(x) = \begin{cases} \int_0^x 1 \mathrm{d}x_1 & 0 \le x \le 1 \\ \int_{x-1}^1 1 \mathrm{d}x_1 & 1 \le x \le 2 \end{cases} = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$$

leading to

$$F_{S_2}(x) = \begin{cases} \frac{x^2}{2} & 0 \le z \le 1 \\ 2z - \frac{z^2}{2} - 1 & 1 \le z \le 2 \end{cases}$$

and $F_{S_2}(1.5) = 0.875$.

Question b) This is a) in example 3 page 379. $P(S_3 \le 1.5) = 0.5$.

Question c) Now using the results of example 3 we get

$$P(S_3 \le 1.1) = \int_0^1 \frac{t^2}{2} \mathrm{d}t + \int_1^{1.1} \left(-t^2 + 3t - \frac{3}{2} \right) \mathrm{d}t = \frac{1}{6} + \left[-\frac{t^3}{3} + \frac{3t^2}{2} - \frac{3t}{2} \right]_{t=1}^{t=1.1} = 0.2213$$

Question d) Using the standard approximation $P(x < S_3 < x + \mathrm{d}x) \tilde{=} f_{S_3}(x) \mathrm{d}x$ we find $P(1 \le S_3 \le 1.001) \tilde{=} \frac{1}{2} \cdot 0.001 = 0.0005$.

IMM - DTU

02405 Probability
2004-4-15
BFN/bfn

Solution for exercise 5.4.3 in Pitman

For $\alpha = \beta$ we have the $Gamma(2, \alpha)$ distribution. We denote the waiting time in queue i by X_i, and the total waiting time by Z.

Question a) The distribution of the total waiting time Z is found using the density convolution formula page 372 for independent variables.

$$f(t) = \int_0^t \alpha e^{-\alpha u} \beta e^{-\beta(t-u)} \mathrm{d}u = \alpha\beta e^{-\beta t} \int_0^t e^{u(\beta-\alpha)} \mathrm{d}u = \frac{\alpha\beta}{\beta - \alpha} \left(e^{-\alpha t} - e^{-\beta t}\right)$$

Question b)

$$E(Z) = E(X_1) + E(X_2) = \frac{1}{\alpha} + \frac{1}{\beta}$$

See e.g. page 480 for the means $E(X_i)$ for the exponential variables .

Question c) Using the independence of X_1 and X_2 we have

$$Var(Z) = Var(X_1) + Var(X_2) = \sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}$$

The last equalit follows from e.g. page 480.

IMM - DTU

02405 Probability
2003-11-20
BFN/bfn

Solution for exercise 5.4.4 in Pitman

Question a) We introduce the random variable X_1 as the time to failure of the first component and X_2 as the additional time to failure of the second component. From the assumption X_1 and X_2 are independent and exponentially distributed with intensity 2λ. The sum of two independent exponentially distributed random variables is gamma(2,2λ) distributed.

Question b) The mean of the gamma distribution is $\frac{2}{2\lambda} = \frac{1}{\lambda}$ and the variance is $\frac{2}{(2\lambda)^2} = \frac{1}{2\lambda^2}$ (page 286,481).

Question c)

$$1 - e^{-2\lambda t_{0.9}}(1 + 2\lambda t_{0.9}) = 0.9$$

$$e^{-2\lambda t_{0.9}}(1 + 2\lambda t_{0.9}) = 0.1$$

IMM - DTU

02405 Probability
2003-11-11
BFN/bfn

Solution for exercise 5.4.6 in Pitman

The argument of example 2 page 375 is easily generalized. Since X_i is gamma(r_i, λ) distributed we can write X_i as

$$X_i = \sum_{j=1}^{r_i} W_{ij}$$

where W_{ij} are independent exponential(λ) variables. Thus

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \sum_{j=1}^{r_i} W_{ij}$$

a sum of $\sum_{i=1}^{n} r_i$ exponential(λ) random variables. The sum is gamma$(\sum_{i=1}^{n} r_i, \lambda)$ distributed.

IMM - DTU

02405 Probability
2003-12-5
BFN/bfn

Solution for exercise 5.4.7 in Pitman

Question a) We apply the method used to derive the distribution of ratios page 382. Such that

$$P(z < Z < z + \mathrm{d}z) = \int_x P(x < X < x + \mathrm{d}x, z < XY < z + \mathrm{d}z)$$

Instead of the cone on page 382 we now have an area between the two curves $xy = z$ and $xy = z + \mathrm{d}z$. Thus we have that the area of the parallelogram for fixed x is approximately equal to

$$\mathrm{d}x \left(\frac{z + \mathrm{d}z}{x} - \frac{z}{x} \right) = \frac{\mathrm{d}x \mathrm{d}z}{x}$$

We get the density of Z by integration over x. Thus

$$f_Z(z) = \int_x \frac{1}{x} f\left(x, \frac{z}{x}\right) \mathrm{d}x$$

Question b) This part follows more or less directly from page 372, such that $Z = X - Y$ has density

$$f_Z(z) = \int_x f(x, x - z) \mathrm{d}x$$

Question c) Introduce $W = 2Y$. The density $f_W(w)$ of W is $\frac{1}{2} f_Y(w)$ from the linear change of variable principle; see e.g. page 333. We now apply the general convolution result page 372 or page 386 for the variables X and W to get

$$f_Z(z) = \int_x \frac{1}{2} f\left(x, \frac{z - x}{2}\right) \mathrm{d}x$$

IMM - DTU

02405 Probability
2006-11-16
BFN/bfn

Solution for exercise 5.4.19 in Pitman

We apply the technique of the proof for the distribution of ratios formula page 382-383. Define $Z = \frac{X}{X+Y}$. The event $z < Z < z + \mathrm{d}z$ occurs whenever Y is between the two lines $\frac{x}{x+y} = z + \mathrm{d}z$ and $\frac{x}{x+y} = z$. We get the length of the vertical side of the rectangle by solving for y in the two equations above. Thus

$$y_2 = \left(\frac{1}{z} - 1\right)x, y_1 = \left(\frac{1}{z+\mathrm{d}z} - 1\right)x, \qquad y_2 - y_1 = \frac{x\mathrm{d}z}{z(z+\mathrm{d}z)} = \tilde{=} \frac{x\mathrm{d}z}{z^2}$$

We have derived a general formula for the density of $Z = \frac{X}{X+Y}$ for non negative X and Y

$$\int_0^\infty \frac{x}{z^2} f_X(x) f_Y\left(\frac{(1-z)x}{z}\right)\mathrm{d}x$$

We now insert the gamma densities of X and Y (see page 481) to get

$$\int_0^\infty \frac{x}{z^2}\lambda\frac{(\lambda x)^{r-1}}{\Gamma(r)}e^{-\lambda x}\lambda\frac{\left(\lambda\frac{(1-z)x}{z}\right)^{s-1}}{\Gamma(s)}e^{-\lambda\frac{(1-z)x}{z}}\mathrm{d}x$$

We simplify to get

$$\frac{1}{z^2\Gamma(r)\Gamma(s)}\left(\frac{1-z}{z}\right)^{s-1}\int_0^\infty \lambda(\lambda x)^{r+s-1}e^{-\lambda\frac{x}{z}}\mathrm{d}x$$

the function under the integral is very close to a gamma density such that with

$$\frac{1}{z^2\Gamma(r)\Gamma(s)}\left(\frac{1-z}{z}\right)^{s-1}\Gamma(r+s)z^{r+s-1}\int_0^\infty \lambda\frac{\left(\frac{(\lambda x)}{z}\right)^{r+s-1}}{\Gamma(r+s)}e^{-\lambda\frac{x}{z}}\mathrm{d}x$$

we get the density of a *gamma* $\left(r+s, \frac{\lambda}{z}\right)$ variable. Thus

$$f_Z(z) = \frac{\Gamma(r+s}{\Gamma(r)\Gamma(s)}(1-z)^{s-1}z^{r+s-(s-1)-2} = \frac{1}{B(r,s)}z^{r-1}(1-z)^{s-1}$$

the density of a *beta*(r,s) random variable.

Independence

Three lines to follow

1. We see directly from the calculations

2. Considering

$$P\left(z < \frac{X}{X+Y} < z + \mathrm{d}z | w < X + Y < w + \mathrm{d}w\right) = \frac{P\left(z < \frac{X}{X+Y} < z + \mathrm{d}z, w < X + Y < w + \mathrm{d}w\right)}{P(w < X + Y < w + \mathrm{d}w)}$$

3. Using the division rule page 425

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 6.1.1 in Pitman

Question a) X is binomially distributed $b\left(3, \frac{1}{2}\right)$.

$$P(X=0) = \frac{1}{8}, P(X=1) = \frac{3}{8}, P(X=2) = \frac{3}{8}, P(X=3) = \frac{1}{8}$$

Question b) We introduce the random variables Z_x with binomial distribution $b\left(3-x, \frac{1}{2}\right)$. We can write $Y = x + Z_x$ for the conditional distribution of Y. For $x = 0$ we get

$$P(Y=0|X=0) = \frac{1}{8}, P(Y=1|X=0) = \frac{3}{8}, P(Y=2|X=0) = \frac{3}{8}, P(Y=3|X=0) = \frac{1}{8}$$

For $x = 1$ we get

$$P(Y=1|X=1) = \frac{1}{4}, P(Y=2|X=1) = \frac{1}{2}, P(Y=3|X=1) = \frac{1}{4}$$

For $x = 2$ we get

$$P(Y=2|X=2) = \frac{1}{2}, P(Y=3|X=2) = \frac{1}{2}$$

For $x = 3$ we get

$$P(Y=3|X=3) = 1$$

Question c) We find $P(X=x, Y=y) = P(X=x)P(Y=y|X=x)$. The distribution table is

X/Y	0	1	2	3
0	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$
1	0	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{3}{32}$
2	0	0	$\frac{3}{16}$	$\frac{3}{16}$
3	0	0	0	$\frac{1}{8}$

Question d) We find the distribution of Y from the distribution table in the previous question

$$P(Y=0) = \frac{1}{64}, P(Y=1) = \frac{9}{64}, P(Y=2) = \frac{27}{64}, P(Y=3) = \frac{27}{64}$$

Question e) Using $P(X=x|Y=y) = \frac{P(X=x,Y=y)}{P(Y=y)}$ we get for $y = 0$

$$P(X=0|Y=0) = 1$$

for $y = 1$

$$P(X = 0|Y = 1) = \frac{1}{3}, P(X = 1|Y = 1) = \frac{2}{3}$$

for $y = 2$

$$P(X = 0|Y = 2) = \frac{1}{9}, P(X = 1|Y = 2) = \frac{4}{9}, P(X = 2|Y = 2) = \frac{4}{9}$$

for $y = 3$

$$P(X = 0|Y = 3) = \frac{1}{27}, P(X = 1|Y = 3) = \frac{2}{9}, P(X = 2|Y = 3) = \frac{4}{9}, P(X = 3|Y = 3) = \frac{8}{27}$$

Question f) Best guess $\hat{X}_y$ of $X|Y = y$

$Y = y$	0	1	2	3
$\hat{X}_y$	0	1	1 or 2	2

Question g)

$$\sum_{y=0}^{3} P(Y = y)P(X = \hat{X}_y) = \frac{1}{64} \cdot 1 + \frac{9}{64} \cdot \frac{2}{3} + \frac{27}{64} \cdot \frac{4}{9} + \frac{27}{64} \cdot \frac{4}{9} = \frac{31}{64}$$

IMM - DTU

02405 Probability
2004-4-17
BFN/bfn

Solution for exercise 6.1.3 in Pitman

Question a) Assuming that the total number of families is n we can deduce that we have $i \cdot P(T = i) \cdot n$ tickets from families with i children, giving a total of $0 \cdot 0.1 \cdot n + 1 \cdot 0.2 \cdot n + 2 \cdot 0.4 \cdot n + 3 \cdot 0.2 \cdot n + 4 \cdot 0.1 \cdot n = 2n$ tickets, $3 \cdot 0.2 \cdot n$ of those from families with 3 children. Using equally likely outcomes (section 1.1) we get $P(U = 3) = 0.3$.

Question b) The probability in question is $P(U = 3, G = 2)$, we find this probability sequentially like in example 1. $P(U = 3, G = 2) = P(U = 3)P(G = 2|U = 3) = 0.3 \cdot \binom{3}{2} 2^{-3} = \frac{9}{80}$

Question c) $P(T = 3, G = 2) = P(T = 3)P(G = 2|T = 3) = 0.2 \cdot \binom{3}{2} 2^{-3} = \frac{3}{40}$

IMM - DTU

02405 Probability
2004-4-17
BFN/bfn

Solution for exercise 6.1.5 in Pitman

Question a) The probability in distribution in question is $P(X_1 = x_1|X_1 + X_2 = n)$. Using the definition of conditioned probabilities

$$P(X_1 = x_1|X_1 + X_2 = n) = \frac{P(X_1 = x_1, X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_1 = x_1, X_2 = n - x_1)}{P(X_1 + X_2 = n)} = \frac{P(X_1 = x_1)P(X_2 = n - x_1)}{P(X_1 + X_2 = n)}$$

where we have used the independence of X_1 and X_2 and the last equality. Now using the Poisson probability expression and the boxed result page 226

$$P(X_1 = x_1|X_1 + X_2 = n) = \frac{\frac{\lambda_1^{x_1}}{x_1!}e^{-\lambda_1}\frac{\lambda_2^{n-x_1}}{(n-x_1)!}e^{-\lambda_2}}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}}$$

$$= \frac{n!}{x_1!(n-x_1)!}\frac{\lambda_1^{x_1}\lambda_2^{n-x_1}}{(\lambda_1+\lambda_2)^n} = \left(\begin{array}{c} n \\ x_1 \end{array}\right) p^{x_1}(1-p)^{n-x_1}$$

with $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

Question b) Let X_i denote the number of eggs laid by insect i. The probability in question is $P(X_1 \geq 90) = P(X_2 \leq 60)$. Now $X_i \in \textit{binomial}\left(150, \frac{1}{2}\right)$. With the normal approximation to the binomial distribution page 99 to get

$$P(X_2 \leq 60) = \Phi\left(\frac{60 + \frac{1}{2} - 150 \cdot \frac{1}{2}}{\frac{1}{2}\sqrt{150}}\right) = \Phi\left(\frac{-29}{\sqrt{150}}\right) = \Phi(-2.37) = 0.0089$$

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 6.1.6 in Pitman

Question a) We recall the definition of conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$, such that

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) \frac{P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m \cap \sum_{i=1}^{m} N_i = n)}{P(\sum_{i=1}^{m} N_i = n)}$$

Now realising that $P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m \cap \sum_{i=1}^{m} N_i = n) = P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m)$ and using the fact that $N = \sum_{i=1}^{m} N_i$ has Poisson distribution with parameter $\lambda = \sum_{i=1}^{m} \lambda_i$ we get

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) = \frac{\prod_{i=1}^{m} \frac{\lambda_i^{n_i}}{n_i!} e^{-\lambda_i}}{\frac{\lambda^{\sum_{i=1}^{m} n_i}}{(\sum_{i=1}^{m} n_i)!} e^{-\lambda}}$$

such that with $n = \sum_{i=1}^{m} n_i$

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) = \frac{n!}{n_1! n_2! \cdots n_m!} \left(\frac{\lambda_1}{\lambda}\right)^{n_1} \left(\frac{\lambda_1}{\lambda}\right)^{n_2} \cdots \left(\frac{\lambda_m}{\lambda}\right)^{n_m}$$

a multinomial distribution (page 155) with probabilities $p_i = \frac{\lambda_i}{\lambda}$.

Question b) Using

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m) = P(N = n) P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n)$$

we see that the N_i's are independent Poisson variables.

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 6.2.1 in Pitman

We have the joint distribution (distribution table) (see e.g. exercise 3.1.4)

$X\backslash Y$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
2	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
3	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
4	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$
6	0	0	0	0	0	$\frac{1}{36}$

We find the conditional distributions to be

$Y:$	1	2	3	4	5	6
$P(X=1\|Y=y)$	1	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$P(X=2\|Y=y)$	0	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$P(X=3\|Y=y)$	0	0	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$P(X=4\|Y=y)$	0	0	0	$\frac{1}{7}$	$\frac{2}{9}$	$\frac{2}{11}$
$P(X=5\|Y=y)$	0	0	0	0	$\frac{1}{9}$	$\frac{2}{11}$
$P(X=6\|Y=y)$	0	0	0	0	1	$\frac{1}{11}$

and

X	$P(Y=1\|1)$	$P(Y=1\|2)$	$P(Y=1\|3)$	$P(Y=1\|4)$	$P(Y=1\|5)$	$P(Y=1\|6)$
1	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$	$\frac{2}{11}$
2	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
3	0	0	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$
4	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
5	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$
6	0	0	0	0	0	1

where we have used the short $P(Y=y|x)$ for $P(Y=y|X=x)$.

Such that

$Y:$	1	2	3	4	5	6
$E(X\|Y=y)$	1	$\frac{4}{3}$	$\frac{9}{5}$	$\frac{16}{7}$	$\frac{25}{9}$	$\frac{36}{11}$

and

X	$E(Y\|X=x)$
1	$\frac{41}{11}$
2	$\frac{38}{9}$
3	$\frac{33}{7}$
4	$\frac{26}{5}$
5	$\frac{17}{3}$
6	6

Solution for exercise 6.2.4 in Pitman

Question a) We first derive

$$E(Y|X=x) = \sum_{y=1}^{x} y \cdot \frac{1}{x} = \frac{1}{x}\sum_{y=1}^{x} y \quad .$$

We have the general formula (from Appendix 2 on sums page 516 (first line of last box))

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

This formula can be derived by induction a by a smart argument. For even n collect in pairs $(1,n)$, $(2,n-2)\ldots$, $(i,n+1-i)\ldots$ and realize that the sum of i and $n+1-i$ is always $n+1$ and that we have $\frac{n}{2}$ of such pairs. The extension for n odd is straightforward. with this result we get

$$E(Y|X=x) = \frac{1}{x}\sum_{y=1}^{x} y = \frac{1}{x}\frac{x(x+1)}{2} = \frac{x+1}{2} \quad .$$

Now

$$E(Y) = E(E(Y|X)) = E\left(\frac{X+1}{2}\right) = \frac{1}{2}E(X) + \frac{1}{2} = \frac{1}{2}\left(\sum_{x=1}^{n} x\frac{1}{n}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\frac{1}{n}\frac{n(n+1)}{2} + \frac{1}{2} = \frac{n+3}{4}$$

Question b)

$$E(Y^2|X=x) = \sum_{y=1}^{x} y^2\frac{1}{x}$$

We have the general formula

$$\sum_{i=1}^{m} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(which we can derive using $E(X^2) = SD(X)^2 + E(X)^2$ for the uniform distribution page 477 or 487). Thus

$$E(Y^2|X=x) = \frac{(x+1)(2x+1)}{6}$$

Now

$$E(Y^2) = E(E(Y^2|X=x)) = \sum_{x=1}^{n} \frac{(x+1)(2x+1)}{6}\frac{1}{n}$$

$$= \left(\frac{1}{3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{n}{6}\right)\frac{1}{n} = \frac{(n+1)(4n+11)+6}{36}$$

Question c) To find $SD(Y)$ we use the computational formula for the variance

$$SD(Y) = \sqrt{E(Y^2) - (E(Y))^2} = \frac{\sqrt{7n^2+6n-13}}{12}$$

after simplifications.

Question d)

$$P(X+Y=2) = P(X+Y=2|X=1)P(X=1) + P(X+Y=2|X \neq 1)P(X \neq 1)$$

$$= P(X+Y=2|X=1)P(X=1) = \frac{1}{n}$$

IMM - DTU

02405 Probability
2003-11-13
BFN/bfn

Solution for exercise 6.2.5 in Pitman

Question a) By the defition of the c.d.f. page 311

$$F(x) = P(X \leq x) = P(X \leq x|A)P(A) + P(X \leq x|A^c)P(A^c)$$

using the rule of averaged conditional probabilities page 41. Now introduce the parameters of the exercise ($P(X \leq x|A) = F_1(x)$ etc.) to get

$$F(x) = p \cdot F_1(x) + (1-p)F_2(x)$$

Question b) We first find the density of X assuming X_1 and X_2 continuous (similar calculations can be made in full generality)

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = p \cdot f_1(x) + (1-p) \cdot f_2(x)$$

with $f_i(x) = \frac{\mathrm{d}F_i(x)}{\mathrm{d}x}, i = 1, 2$. Now (see e.g. page 261 top)

$$E(X) = \int x \cdot f(x)\mathrm{d}x = \int x(p \cdot f_1(x) + (1-p) \cdot f_2(x))\mathrm{d}x$$

using the linearity of the integral we get

$$= p \int x \cdot f_1(x)\mathrm{d}x + (1-p) \int x \cdot f_2(x)\mathrm{d}x = p \cdot E(X_1) + (1-p) \cdot E(X_2)$$

Question c) We first note that we can derive $E(X^2)$ in a similar way, thus

$$E(X^2) = p \cdot E(X_1^2) + (1-p) \cdot E(X_2^2) = p \cdot (Var(X_1) + E(X_1)^2) + (1-p) \cdot (Var(X_2) + E(X_2)^2)$$

where we have used the computational formula for the variance e.g. page 261. Applying this formula once more we get

$$Var(X) = E(X^2) - E(X)^2$$

$$= p \cdot (Var(X_1) + E(X_1)^2) + (1-p) \cdot (Var(X_2) + E(X_2)^2) - (p \cdot E(X_1) + (1-p) \cdot E(X_2))^2$$
$$= p \cdot Var(X_1) + (1-p) \cdot Var(X_2) + p(1-p)(E(X_1) - E(X_2))^2$$

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 6.2.18 in Pitman

By definition

$$Var(Y) = \sum_y (y - E(Y))^2 f(y) = \sum_y (y - E(Y))^2 \sum_x f(x,y) = \sum_x \sum_y (y - E(Y))^2 f(x,y)$$

We now apply the crucial idea of adding 0 in the form of $E(Y|x) - E(Y|x)$ inside the brackets.

$$Var(Y) = \sum_x \sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 f(x,y)$$

Next we multiply with one in the form of $\frac{f(x)}{f(x)}$

$$Var(Y) = \sum_x \sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 \frac{f(x,y)}{f(x)} f(x)$$

By definition $f_Y(y|x) = \frac{f(x,y)}{f(x)}$ thus

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

Expanding the square sum we get

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x))^2 + (E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

since $\sum_y (y - E(Y|x)) = 0$. Now

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x))^2 f_Y(y|x) \right] f(x) + \sum_x \left[\sum_y (E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

the inner part of the first term is $Var(Y|X = x)$ while the inner part of the second term is constant. Thus

$$Var(Y) = \sum_x Var(Y|X = x) f(x) + \sum_x (E(Y|x) - E(Y))^2 f(x)$$

leading to the stated equation

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

an important and very useful result that is also valid for continuous and mixed distributions. Mixed distributions are distributions that are neither discrete nor continuous.

IMM - DTU

02405 Probability
2004-4-17
BFN/bfn

Solution for exercise 6.3.1 in Pitman

Use the boxed formula at the bottom of page 417 to get

$$P(A) = \int_0^1 x^2 f_x(x) \mathrm{d}x = \int_0^1 x^2 \mathrm{d}x = \frac{1}{3}$$

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 6.3.5 in Pitman

We note that Y for given $X = x$ is uniformly distributed, on $1 + x$ for $-1 < x < 0$ and on $1 - x$ for $0 < x < 1$. Thus

$$F(y|x) = P(Y \leq y|X = x) = \frac{y}{1 - |x|}, 0 < y < 1 - |x|$$

Question a) We have $P\left(Y \geq \frac{1}{2}|X = x\right) = 1 - F\left(\frac{1}{2}|x\right)$

Question b) We have $P\left(Y \leq \frac{1}{2}|X = x\right) = F\left(\frac{1}{2}|x\right)$

Question c) Since Y for given $X = x$ is uniformly distributed we can apply results for the uniform distribution, see e.g. the distribution summary page 477 or 487. We get

$$E(Y|X = x) = \frac{1 - |x|}{2}$$

Question c) Similarly

$$Var(Y|X = x) = \frac{(1 - |x|)^2}{12}$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 6.3.8 in Pitman

Question a) Since $Y|X=x$ is $bin(5,x)$ distributed we immediately have

$$E(Y|X=x) = 5 \cdot x, E(Y^2|X=x) = 5x(1-x) + 25x^2 = 5x(1+4x)$$

where we have used the computational formula for the variance to get $E(Y^2|X=x)$. Now using the boxed result page 403

$$E(Y) = E(E(Y|X)) = \int_0^1 5x\mathrm{d}x = \frac{5}{2}$$

and (once again using page 403)

$$E(Y^2) = E(E(Y^2|X)) = \int_0^1 (5x + 20x^2)\mathrm{d}x = \frac{55}{6}$$

Question b)

$$P(Y=y, x<X<x+\mathrm{d}x) = f(x)\mathrm{d}xP(Y=y|x<X<x+\mathrm{d}x) = 1 \cdot \mathrm{d}x \left(\begin{array}{c} 5 \\ y \end{array}\right) x^y(1-x)^{5-y}$$

Question c) To find the density we consider

$$P(x<X<x+\mathrm{d}x|Y=y) = \frac{P(Y=y, x<X<x+\mathrm{d}x)}{P(Y=y)}$$

The probability $P(Y=y)$ in the denominator is found by

$$P(Y=y) = \int_0^1 1 \cdot \left(\begin{array}{c} 5 \\ y \end{array}\right) x^y(1-x)^{5-y}\mathrm{d}x$$

Such that

$$P(x<X<x+\mathrm{d}x|Y=y) = f(x|Y=y)\mathrm{d}x = \frac{\left(\begin{array}{c} 5 \\ y \end{array}\right) x^y(1-x)^{5-y}\mathrm{d}x}{\int_0^1 1 \cdot \left(\begin{array}{c} 5 \\ y \end{array}\right) x^y(1-x)^{5-y}\mathrm{d}x}$$

which we reckognize as a $beta(y+1, 6-y)$ distribution (page 478).

IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 6.3.14 in Pitman

We have immediately

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i}$$

The posterior density of p given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{f(p; X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}$$

$$= \frac{f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|p)f(p)}{\int_0^1 f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|p)f(p)\mathrm{d}p}$$

Inserting the previous result to get

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i}f(p)}{\int_0^1 p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i}f(p)\mathrm{d}p}$$

which only dependes on the X_i's through their sum. Introducing $S_n = \sum_{i=1}^n X_i$ we rewrite

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{p^{S_n}(1-p)^{n-S_n}f(p)}{\int_0^1 p^{S_n}(1-p)^{n-S_n}f(p)\mathrm{d}p}$$

We note that if the prior density of p $f(p)$ is a $beta(r, s)$ distribution, then the posterior distribution is a $beta(r + S_n, s + n - S_n)$ distribution.

Solution for exercise 6.4.1 in Pitman

Question a) From the definition of conditional proability we have $P(A|B) = \frac{A\cap B}{P(B)}$. Now from inclusion-exclusion e.g. page 22 we have $P(A\cap B) = P(A) + P(B) - P(A\cup B)$. Thus

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A) + P(B) - P(A\cup B)}{P(B)} = \frac{1}{2}$$

Question b) Since $P(A)P(B) = 0.12 < 0.2 = P(A\cap B)$ we conclude that A and B are positive depdendent (page 431).

Question c) Using $B = (A\cap B)\cup(A^c\cap B)$ we find $P(A^c\cap B) = P(B) - P(A\cap B) = 0.2$

Question d) We find for the Bernoulli distribution which is the binomial distribution with $n = 1$ (e.g. page 479) $\sigma_X = \sqrt{0.3\cdot 0.7}$ and $\sigma_Y = \sqrt{0.4\cdot 0.6}$. Further $E(XY) = P(I_A\cdot I_B = 1) = P(A\cap B) = 0.2$. Using $Cov(X,Y) = E(XY) - E(X)E(Y)$ page 430 and the correlation definition page 432 we get

$$Corr(X,Y) = \frac{0.2 - 0.12}{\sqrt{0.21\cdot 0.24}} = 0.356$$

Solution for exercise 6.4.2 in Pitman

From $P(A) = P(A|B)P(B)+P(A|B^c)P(B^c)$ we realize that $P(A)$ is a weighted average of $P(A|B)$ and $P(A|B^c)$, thus one and only one of

1. $P(A|B) = P(A) = P(A|B^c)$
2. $P(A|B) < P(A) < P(A|B^c)$
3. $P(A|B) > P(A) > P(A|B^c)$

is true.

Question a) Obvious from page 42.

Question b) We have

$$Cov(I_A, I_B) = P(A \cap B) - P(A)P(B) = P(A|B)P(B) - P(A)P(B) = (P(A|B) - P(A))P(B) > 0$$

Question c) As for c) interchanging the roles of B and B^c.

Question d) Once again obvious from page 42.

Question e) We have $(P(A|B) - P(A))P(B) > 0$ since A and B are positively dependent. We dedeuce that $P(A|B) > P(A)$ implying $P(A|B) > P(A|B^c)$

Question f) As for e).

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for exercise 6.4.5 in Pitman

Question a) We calculate the covariance of X and Y using the definition page 630.

$$Cov(X,Y) = E(XY) - E(X)E(Y) = E(XY)$$

since $E(X) = 0$ We calculate

$$E(XY) = E(X^3) = \int_{-1}^{1} x^3 \frac{1}{2} \mathrm{d}x = 0$$

thus X and Y are uncorrelated.

Question b) We have

$$P\left(Y > \frac{1}{4} \middle| |X| > \frac{1}{2}\right) = 1 \neq P\left(Y > \frac{1}{4}\right)$$

thus X and Y are *not* independent.

IMM - DTU

02405 Probability
2003-11-11
BFN/bfn

Solution for exercise 6.4.6 in Pitman

X and Y are clearly not indpendent.

$P(X=0|Y=12) = P(X_1-X_2=0|X_1+X_2=12) = 1 \neq P(X_1-X_2=0) = P(X=0)$

However, X and Y are uncorrelated:

$$Cov(X,Y) = E((X-E(X))(Y-E(Y))) = E(XY) - E(X)E(Y) = E(XY)$$
$$= E((X_1-X_2)(X_1+X_2)) = E(X_1^2 - X_2^2) = E(X_1^2) - E(X_2^2) = 0$$

using the definition of covariance page 630

IMM - DTU

02405 Probability
2004-5-13
BFN/bfn

Solution for exercise 6.4.7 in Pitman

Question a)

X_2	X_3	$X_2 + X_3$	$X_2 - X_3$	Probability
0	0	0	0	$\frac{1}{3}$
0	1	1	-1	$\frac{1}{6}$
1	0	1	1	$\frac{1}{3}$
1	1	2	0	$\frac{1}{6}$

$X_2 + X_3$ / $X_2 - X_3$	-1	0	1
0	0	$\frac{1}{3}$	0
1	$\frac{1}{6}$	0	$\frac{1}{3}$
2	0	$\frac{1}{6}$	0

Question b) With $Z_2 = X_2 - X_3$ we get $E((X_2 - X_3)^3) = E(Z_2^3) = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6}$.

Question c) X_2 and X_3 are independent thus uncorrelated. The new variables $Z_1 = X_2 + X_3$ and $Z_2 = X_2 - X_3$ are correlated. $E(Z_1 Z_2) = E(X_2^2) - E(X_3^2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \neq \frac{5}{6}\frac{1}{6} = E(Z_1)E(Z_2)$

IMM - DTU

02405 Probability
2006-11-28
BFN/bfn

Solution for exercise 6.5.1 in Pitman

Question a) We define U to be a students score on the PSAT test and V to be the score of the same student on the PSAT test. The pair (U, V) follows a general bivariate normal distribution as given in the box at the bottom of page 454. The probability in question is $P(V > 1300|U = 1000)$ which we rewrite

$$P(V > 1300|U = 1000) = P\left(\frac{V-1300}{90}\middle|\frac{U-1200}{100} = -2\right)$$

Now using the definition on page 454 together with the definition of the standard bivariate normal distribution page 451 we get

$$P(0.6 \cdot X + \sqrt{1-0.6^2}Z > 0|X = -2) = P\left(Z > \frac{1.2}{0.8}\right) = 1 - \Phi(1.5) = 0.0668$$

Question b) The solution to this question is closely related to the method of Example 2 page 457. First we realize that we can consider standard normal variates. Using the notation of the previous question we formulate the problem as

$$P(Y > 0|X < 0) = P(0.6 \cdot X + 0.8 \cdot Z > 0|X < 0) = \frac{P(0.6 \cdot X + 0.8 \cdot Z > 0, X < 0)}{P(X < 0)}$$

Now using the rotational symmetry we see that $P(0.6 \cdot X + 0.8 \cdot Z > 0, X < 0) = \frac{90 - \tan^{-1}\left(\frac{3}{4}\right)}{360} = 0.14758$. Finally $P(Y > 0|X < 0) = \frac{0.14758}{0.5} = 0.2952$.

Question c) We formulate the question using the notation of question a) as

$$P(V - U > 50)$$

we get

$$P(1300 + 90 \cdot Y - (1200 + 100 \cdot X) > 50)$$

$$= P(1300 + 90 \cdot (\rho X + \sqrt{1-\rho^2}Z) - (1200 + 100 \cdot X) > 50)$$

$$= P(72Z - 46X \geq -50) = 1 - \Phi\left(-\frac{50}{\sqrt{46^2 + 72^2}}\right) = \Phi(0.585) = 0.72$$

IMM - DTU

02405 Probability
2005-5-12
BFN/bfn

Solution for exercise 6.5.2 in Pitman

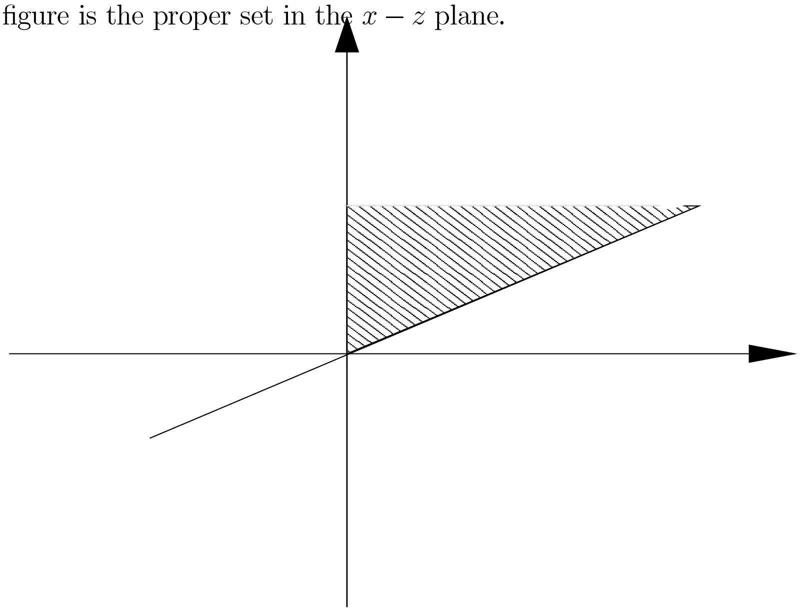
First we realize that we can consider standard normal variates ignoring specific parameter values. For a bivariate standard normal distribution (X, Y) with correlation 0.5 we need to solve for

$$P(Y > X | X > 0) = \frac{P(Y > X > 0)}{P(X > 0)}$$

Now applying the technique of example 2 page 454 we get

$$\frac{P(Y > X > 0)}{P(X > 0)} = \frac{P(\rho \cdot X + \sqrt{1-\rho^2} Z > X > 0)}{P(X > 0)} = \frac{P\left(Z > \frac{1-\rho}{\sqrt{1-\rho^2}} X > 0\right)}{P(X > 0)}$$

where (X, Z) are bivariate normal and independent.The shaded area in the figure is the proper set in the $x - z$ plane.



Using the rotational symmetry we find the probability to be $\frac{60}{180} = \frac{1}{3}$ or $\frac{\frac{60}{360}}{\frac{1}{2}} = \frac{1}{3}$.

IMM - DTU

02405 Probability
2003-11-11
BFN/bfn

Solution for exercise 6.5.4 in Pitman

Question a) We have from the boxed result page 363 that $X + 2Y$ is normally distributed with mean $\mu = 0 + 2 \cdot 0 = 0$ and variance $\sigma^2 = 1 + 4 \cdot 1 = 5$. We now evaluate

$$P(X + 2Y \leq 3) = P\left(\frac{X + 2Y}{\sqrt{5}} \leq \frac{3}{\sqrt{5}}\right) = \Phi\left(\frac{3}{\sqrt{5}}\right) = \Phi(1.34) = 0.9099$$

Question b) We have from the boxed result page 451

$$Y = \frac{1}{2}X + \sqrt{1 - \frac{1}{4}}Z$$

where X and Z are indpendent standard normal variables. Thus

$$X + 2Y = 2X + \sqrt{3}Z$$

This is the sum of two independent normal variables which itself is $Normal(0, 2^2 + \sqrt{3}^2)$ distributed. Thus

$$P(X + 2Y \leq 3) = \Phi\left(\frac{3}{\sqrt{7}}\right) = \Phi(1.13) = 0.8708$$

IMM - DTU

02405 Probability
2004-5-13
BFN/bfn

Solution for exercise 6.5.6 in Pitman

Question a)

$$P(X > kY) = P(X - kY > 0)$$

From the boxed result page 363 we know that $Z = X - kY$ is $normal(0, 1 + k^2)$ distributed, thus $P(X - kY > 0) = \frac{1}{2}$.

Question b) Arguing along the same lines we find $P(U > kV) = \frac{1}{2}$.

Question c)

$$P(U^2 + V^2 < 1) = P(3X^2 + Y^2 + 2\sqrt{3}XY + X^2 + 3Y^2 - 2\sqrt{3}XY < 1)$$

$$= P\left(X^2 + Y^2 < \frac{1}{4}\right) = 1 - e^{-\frac{1}{8}} = 0.118$$

where we have used $X^2 + Y^2 \in exponential(0.5)$ in the last equality (page 360, 364-366, 485).

Question d)

$$X = v + \sqrt{3}Y \in normal(v, 3)$$

IMM - DTU

02405 Probability
2003-12-12
BFN/bfn

Solution for exercise 6.5.10 in Pitman

Question a) We first note from page that since V are W are bivariate normal, then

$$X = \frac{V - \mu_V}{\sigma_V} \qquad Y = \frac{W - \mu_W}{\sigma_W}$$

are bivariate standardized normal. From page we have that we can write

$$Y = \rho X + \sqrt{1-\rho^2} Z$$

where X and Z are standardized independent normal variables. Thus any linear combination of V and W will be a linear combination of X and Z. We know from chapter 5. that such a combination is a normal variable. After some tedious calculations we find the actual linear combinations to be

$$aV + bW = a\mu_V + b\mu_W + (a\sigma_V + b\rho\sigma_W)X + b\sigma_2\sqrt{1-\rho^2}Z$$

and

$$cV + dW = c\mu_V + d\mu_W + (c\sigma_V + d\rho\sigma_W)X + d\sigma_2\sqrt{1-\rho^2}Z$$

Such that $(aV + bW \in normal(a\mu_V + b\mu_W, a^2\sigma_V^2 + b^2\sigma_W^2 + 2ab\rho\sigma_V\sigma_W)$ and $(cV + dW \in normal(c\mu_V + d\mu_W, c^2\sigma_V^2 + d^2\sigma_W^2 + 2cd\rho\sigma_V\sigma_W)$.

Question b) We have from question a) that

$$V_1 = aV + bW = \mu_1 + \gamma_{11}X + \gamma_{12}Z \qquad W_1 = cV + dW = \mu_2 + \gamma_{21}X + \gamma_{22}Z$$

for some appropriate constants. We can rewrite these expressions to get

$$\frac{V_1 - \mu_1}{\sqrt{\gamma_{11}^2 + \gamma_{12}^2}} = \frac{\gamma_{11}X + \gamma_{12}Z}{\sqrt{\gamma_{11}^2 + \gamma_{12}^2}} = X_1 \qquad \frac{W_1 - \mu_2}{\sqrt{\gamma_{21}^2 + \gamma_{22}^2}} = \frac{\gamma_{21}X + \gamma_{22}Z}{\sqrt{\gamma_{21}^2 + \gamma_{22}^2}} = Y_1$$

such that X_1 and Y_1 are standard normal variables. We see that with some effort we would be able to write

$$Y_1 = \rho_1 X_1 + \sqrt{1-\rho_1^2} Z_1$$

and we conclude from page 454 that V_1 and W_2 are bivariate normal variables.

Question c) We find the parameters using standard results for mean and variance

$$\mu_1 = E(aV + bW) = a\mu_V + b\mu_W \qquad \mu_2 = E(cV + dW) = c\mu_V + b\mu_W$$
$$\sigma_1^2 = a^2\sigma_V^2 + b^2\sigma_W^2 + 2ab\rho\sigma_V\sigma_W \qquad \sigma_2^2 = c^2\sigma_V^2 + d^2\sigma_W^2 + 2cd\rho\sigma_V\sigma_W$$

We find the covariance from

$$E((aV + bW - (a\mu_V + b\mu_W))(cV + dW - (c\mu_V + d\mu_W)))$$
$$= E[(a(V - \mu_V) + b(W - b\mu_W))(c(V - \mu_V) + d(W - \mu_W))]$$

etc

IMM - DTU

02405 Probability
2003-9-11
BFN/bfn

Solution for review exercise 1 (chapter 1) in Pitman

Define the events

$B0$: 0 defective items in box

$B1$: 1 defective item in box

$B2$: 2 defective items in box

I : Item picked at random defective

The question can be stated formally(mathematically) as

$$P(B2|I) = \frac{P(I|B2)P(B2)}{P(I|B0)P(B0) + P(I|B1)P(B1) + P(I|B2)P(B2)} = \frac{1 \cdot 0.03}{0 \cdot 0.92 + 0.5 \cdot 0.05 + 1 \cdot 0.03} = \frac{6}{11}$$

IMM - DTU

02405 Probability
2004-2-10
BFN/bfn

Solution for review exercise 3 (chapter 1) in Pitman

The outcomes of the experiment are $HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$ taking the sequence into account, assuming that these 8 outcomes are equally likely we see that the probability that the coin lands the same way at all three tosses is $\frac{1}{4}$. The flaw in the argument is the lack of independence. We use knowledge obtained from the experiment to choose the tosses which satisfy the requirement that the coin landed the same way at these specific tosses. It is thus less likely that the toss not chosen in the selection procedure had the same result, as one can verify by examining the outcome space.

IMM - DTU

02405 Probability
2004-10-16
BFN/bfn

Solution for review exercise 10 (chapter 1) in Pitman

We define the events

E_k Exactly k blood types are represented

Ai i persons have blood type A

Bi i persons have blood type B

Ci i persons have blood type C

Di i persons have blood type D

Question a)

$$P(E_2) = P(A_2) + P(B_2) + P(C_2) + P(D_2) = p_a^2 + p_b^2 + p_c^2 + p_d^2 = 0.3816$$

Question b) We have $p(k) = P(E_k)$. By combinatorial considerations we can show

$$P(A_{i_1} \cap B_{i_2} \cap C_{i_3} \cap D_{i_4}) = \frac{(i_1 + i_2 + i_3 + i_4)!}{i_1! i_2! i_3! i_4!} p_a^{i_1} p_b^{i_2} p_c^{i_3} p_d^{i_4}$$

with $i_1 + i_2 + i_3 + i_4 = 4$, in our case. We have to sum over the appropriate values of (i_1, i_2, i_3, i_4).

It is doable but much more cumbersome to use basic rules. We get

$$p(1) = 0.0687 \qquad p(2) = 0.5973 \qquad p(3) = 0.3163 \qquad p(4) = 0.0177$$

$$p(1) = P(E_1) = P(A_4) + P(B_4) + P(C_4) + P(D_4) = p_a^4 + p_b^4 + p_c^4 + p_d^4 = 0.0687$$
$$p(4) = P(E_4) = P(A_1 \cap B_1 \cap C_1 \cap D_1) = 24 p_a p_b p_c p_d = 0.0177$$

To calculate $p(3) = P(E_3)$ we use the law of averaged conditional probabilities

$$p(3) = P(E_3) = \sum_{i=0}^{4} P(E_3|A_i) P(A_i).$$

We immediately have

$$P(E_3|A_4) = P(E_3|A_3) = 0$$

To establish $P(E_3|A_2)$ we argue

$$P(E_3|A_2) = P(B_1 \cap C_1|A_2) + P(B_1 \cap D_1|A_2) + P(C_1 \cap D_1|A_2) = \frac{p_b p_c + p_b p_d + p_c p_d}{(1-p_a)^2}$$

further

$$P(E_3|A_0) = P(B_2 \cap C_1 \cap D_1|A_0) + P(B_1 \cap C_2 \cap D_1|A_0) + P(B_1 \cap C_1 \cap D_2|A_0) = \frac{4 p_b p_c p_d (p_b + p_c + p_d}{(1-p_a)^4}$$

To evaluate $P(E3|A_1)$ we use the law of averaged conditional probability once more (see Review Exercise 1.13)

$$P(E3|A1) = \sum_{i=1}^{4} P(E3|A_1 \cap B_i) P(B_i|A_1)$$

with

$$P(E3|A_1 \cap B_0) = \frac{3 p_c p_d (p_c + p_d)}{(1 - p_a - p_b)^3}$$

$$P(E3|A_1 \cap B_1) = \frac{p_c^2 + p_d^2}{(1 - p_a - p_b)^2}$$

$$P(E3|A_1 \cap B_2) = \frac{p_c + p_d}{1 - p_a - p_b}$$

$$P(E3|A_1 \cap B_3) = 0$$

and we get

$$P(E3|A1) = \frac{3 p_c p_d (p_c + p_d)}{(1 - p_a - p_b)^3} \left(\frac{1 - p_a - p_b}{1 - p_a} \right)^3 + \frac{p_c^2 + p_d^2}{(1 - p_a - p_b)^2}$$

IMM - DTU

02405 Probability
2003-10-2
BFN/bfn

Solution for review exercise 15 (chapter 1) in Pitman

Define the events Bi that box i is chosen, and the event G that a gold coin is found. We have

$$P(G|B1) = 1, P(G|B2) = 0, P(G|B3) = \frac{1}{2}$$

We want to find $P(B1|G)$. The probability is found using Baye's rule (p.49)

$$P(B1|G) = \frac{P(G|B1)P(B1)}{P(G|B1)P(B1) + P(G|B2)P(B2) + P(G|B3)P(B3)} = \frac{2}{3}$$

IMM - DTU

02405 Probability
2003-10-12
BFN/bfn

Solution for review exercise 13 (chapter 2) in Pitman

The probability that the manufacturer will have to replace a packet is

$$P(\text{replace}) = \sum_{i=3}^{50} \binom{50}{i} 0.01^i 0.99^{50-i} = 1 - \sum_{i=0}^{2} \binom{50}{i} 0.01^i 0.99^{50-i}$$

$$= 0.99^{50} \left(1 + \frac{0.01}{0.99} \cdot 50 \left(1 + \frac{0.01}{0.99} \cdot \frac{49}{2}\right)\right) = 0.0138$$

Pitman claims this probability to be 0.0144. We evaluate the second probability using the Normal approximation to the Binomial distribution. Let X denote the number of packets the manufacturer has to replace. The random variable X follows a Binomial distribution with $n = 4000$ and $p =$. We can evaluate the probability using the normal approximation.

$$P(X > 40) = 1 - P(X \leq 40) \simeq 1 - \Phi\left(\frac{40 + \frac{1}{2} - 4000 \cdot 0.0138}{\sqrt{4000 \cdot 0.0138 \cdot 0.9862}}\right)$$

$$1 - \Phi\left(\frac{-14.77}{7.38}\right) = 1 - \Phi(-2.00) = 0.9772$$

Slightly different from Pitman's result due to the difference above.

IMM - DTU

02405 Probability
2003-10-15
BFN/bfn

Solution for review exercise 20 (chapter 2) in Pitman

We define the events Bi that exactly i bits are transmitted correctly and the event W that a word is transmitted correctly.

Question a) We can express the event W in terms of the Bi's by $W = \cup_{i=n-k}^{n} Bi$. The events Bi are mutually exclusive such that using the addition rule page 21 we get

$$P(W) = P\left(\cup_{i=n-k}^{n} Bi\right) = \sum_{i=n-k}^{n} P(Bi)$$

Now the probabilities $P(Bi)$ are given by the Binomial distribution page 81 and page 479, so

$$P(W) = \sum_{i=n-k}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i}$$

Question b)

$$P(W) = 0.99^8 \left((1 + 8\frac{0.01}{0.99} \left(1 + \frac{7}{2}\frac{0.01}{0.99} \right) \right) = 0.999946$$

IMM - DTU

02405 Probability
2003-10-12
BFN/bfn

Solution for review exercise 25 (chapter 2) in Pitman

Question a) We define the events Ai that player A wins in i sets. We have immediately

$$P(A3) = p^3$$

Correspondingly, player A can win in 4 sets if he wins 2 out of the first 3 and the 4'th.

$$P(A4) = p \cdot p \cdot q \cdot p + p \cdot q \cdot p \cdot p + q \cdot p \cdot p \cdot p = 3p^3 q$$

similary we find

$$P(A5) = 6p^3 q^2$$

Question b) The event A (player A wins) is $A = A1 \cup A2 \cup A3$. The events Ai are mutually exclusive and we get

$$P(A) = P(A1 \cup A2 \cup A3) = P(A1) + P(A2) + P(A3) = p^3(1 + 3q + 6q^2)$$

Question c) The question can be reformulated as

$$P(A3|A) = \frac{P(A3 \cap A)}{P(A)} = \frac{1}{1 + 3q + 6q^2}$$

using the general formula for conditional probability p.36.

Question d)

$$\frac{3}{8}$$

Question e) Pitman suggests no, which is reasonable. However, the way to assess whether we can assume independence or not would be to analyze the distribution of the number of sets played in a large number of matches.

IMM - DTU

02405 Probability
2006-11-1
BFN/bfn

Solution for review exercise 2.28 (chapter 2) in Pitman

Question a) We define the events A_i that person i receives a correct letter. Each person has a probability of $\frac{1}{n}$ of receiving the correct letter. Thus we have $P(A_i) = \frac{1}{n}$. From the multiplication rule (boxed result at the top of page 37) we have $P(A_i \cap A_j) = P(A_i)P(A_j|A_i)$. Knowing that at person got the right letter, we can conceptually remove this letter from the considerations and rethink the problem with $n-1$ letters. Thus the conditional probability $P(A_j|A_i)$ is $\frac{1}{n-1}$. Generally we can write $P(\cap_{i=1}^{k} A_i) = P(A_1)P(A_2|A_1)\cdots P(A_k|A_1 \cap A_2 \cdots \cap A_{k-1}) = \frac{1}{n\cdot(n-1)\ldots(n-k+1)} = \frac{(n-k)!}{n!}$. The event that at least one letter is correctly addressed is the union of all the events A_i. From exclusion-inclusion we get

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots (-1)^{n-1} P\left(\cap_{i=1}^{n} A_i\right)$$

$$= \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} \frac{(n-i)!}{n!} = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!}$$

Question b) The sum in question a) is close to the first n terms of the Taylor expansion of e^{-1}. Thus approximately for large n

$$P(\cup_{i=1}^{n} A_i) \cong 1 - \frac{1}{e}$$

IMM - DTU

02405 Probability
2003-10-3
BFN/bfn

Solution for review exercise 33 (chapter 2) in Pitman

Question a) Throw the coin twice, repeat if you get two heads. The event with probability $\frac{1}{3}$ now occurs if you got two tails, otherwise the complentary event occurred.

Question b) Throw the coin twice, repeat until you get one head and one tail. Then use HT or TH as the two possibilities.

IMM - DTU

02405 Probability
2003-10-15
BFN/bfn

Solution for review exercise 35 (chapter 2) in Pitman

Question a)

$$\sum_{i=20}^{35} \binom{1000}{i} \left(\frac{1}{38}\right)^i \left(\frac{37}{38}\right)^{1000-i}$$

Question b) The standard deviation $\sqrt{1000\frac{1}{38}\frac{37}{38}} \simeq 5.1$ is acceptable for the Normal approximation.

$$\Phi\left(\frac{35+\frac{1}{2}-1000\frac{1}{38}}{\sqrt{1000\frac{1}{38}\frac{37}{38}}}\right) - \Phi\left(\frac{20-\frac{1}{2}-\frac{1000}{38}}{\sqrt{1000\frac{1}{38}\frac{37}{38}}}\right) = \Phi(1.814) - \Phi(-1.346) = 0.8764$$

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for review exercise 19 (chapter 3) in Pitman

Question a)

$$P(Y \geq X) = \sum_{x=0}^{\infty} P(X = x)P(Y \geq X | X = x)$$

now X and Y are independent such that

$$P(Y \geq X) = \sum_{x=0}^{\infty} P(X = x)P(Y \geq x)$$

There is a convenient formula for the tail probabilities of a geometric distribution, see eg. page 482. We need to adjust this result to the present case of a geometric distribution with range $0, 1, \ldots$ (counting only failures), such that $P(Y \geq x) = (1-p)^x$. We now insert this result and the Poisson densities to get

$$P(Y \geq X) = \sum_{x=0}^{\infty} \frac{\mu^x}{x!} e^{-\mu}(1-p)^x = e^{-\mu} e^{\mu(1-p)} = e^{-\mu p}$$

where we have used the exponential series $\sum_{x=0}^{\infty} \frac{(\mu(1-p))^x}{x!} = e^{\mu(1-p)}$.

Question b)

$$e^{-\mu p} = e^{-\frac{1}{2}} = 0.6065$$

IMM - DTU

02405 Probability
2006-3-9
BFN/bfn

Solution for review exercise 24 (chapter 3) in Pitman

Question a) Following the hint, we write down the permutations of $\{1, 2, 3\}$

$X = x$	$Y = y$	$Z = z$	$I(X > Y)$	$I(Y > Z)$	$I(Z > X)$
1	2	3	0	0	1
1	3	2	0	1	1
2	1	3	1	0	1
2	3	1	0	1	0
3	1	2	1	0	0
3	2	1	1	1	0

By picking the three sequences $\{1, 3, 2\}, \{2, 1, 3\}, \{3, 2, 1\}$ and assigning equal probability $\left(\frac{1}{3}\right)$ to each of them we get

$$P(X > Y) = P((X, Y, Z) \in \{\{2, 1, 3\}, \{3, 2, 1\}\} = \frac{2}{3}, P(Y > Z) = \frac{2}{3}, P(Z > X) = \frac{2}{3}$$

as we wanted to show.

Question b)

$$P(X > Y)+P(Y > Z)+P(Z > X) = E(I_{X>Y})+E(I_{Y>Z})+E(I_{Z>X}) = E(I_{X>Y}+I_{Y>Z}+I_{Z>X})$$

The sum of $I_{X>Y} + I_{Y>Z} + I_{Z>Z}$ can not be greater than 2, thus the smallest of the three probabilities $P(X > Y), P(Y > Z), P(Z > X)$ can not exceed $\frac{2}{3}$.

Question c) By a proper mixture of the preferences A for B, B for C, and C for A. Assume that the people in the survey are equally divided among the three possible rankings.

Question d) We assign equal probability $\left(\frac{1}{n}\right)$ to the permuations

$$\{n, n-1, \ldots, 2, 1\}, \{1, n, n-1, \ldots, 3, 2\}, \ldots, \{n-1, n-2, \ldots, 1, n\}$$

In the sequences $X_1, X_2, \ldots, X_n$, only one of the relations $X_i > X_{i+1}$ will be violoated. (for $i = n$ the relation is $X_n > X_1$).

Question e)

$$P(X > Y) = p_1 + (1-p_1)(1-p_2), P(Y > Z) = p_2, P(Z > X) = 1 - p_1$$

We can achieve $p = P(X > Y) = P(Y > Z) = P(Z > X)$ for $p_1 = \frac{3-\sqrt{5}}{2}$ and $p_2 = \frac{\sqrt{5}-1}{2}$.

IMM - DTU

02405 Probability
2003-10-13
BFN/bfn

Solution for review exercise 25 (chapter 3) in Pitman

Question a) The joint distribution of (Y_1, Y_2) is given by

Y_1/Y_2	0	1	2
0	$\frac{9}{36}$	$\frac{6}{36}$	$\frac{3}{36}$
1	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$
2	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

as a check we verify that the sum of all entries in the table is 1. We derive the distribution of $Y_1 + Y_2$

$Y_1 + Y_2 = i$	0	1	2	3	4
$P(Y_1 + Y_2 = i)$	$\frac{9}{36}$	$\frac{12}{36}$	$\frac{10}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Question b)

$$E(3Y_1+2Y_2) = E(3Y_1)+E(2Y_2) = 3E(Y_1)+2E(Y_2) = 5E(Y_1) = 5\left(0\cdot\frac{1}{2}+1\cdot\frac{1}{3}+2\cdot\frac{1}{6}\right) = \frac{10}{3}$$

The first equality is true due to the addition rule for expectations (page 181), the second equality is true due to the result for linear functions of random variables page 175 b., the third equality is true since Y_1 and Y_2 has the same distribution, and the fourth equality is obtained from the definition of the mean see page 181.

Question c)

$$f(x) = \begin{cases} 0 & \text{for } X \le 3 \\ 1 & \text{for } 4 \le X \le 5 \\ 2 & \text{for } X = 6 \end{cases}$$

or something similar.

IMM - DTU

02405 Probability
2003-11-6
BFN/bfn

Solution for review exercise 29 (chapter 3) in Pitman

Question a) We note that the probability does not depend on the ordering, i.e. the probability of a certain sequence depends on the number of 1's among the X_i's not on the ordering.

$$\frac{\prod_{j=0}^{k-1}(b+jd)\prod_{j=0}^{n-k-1}(w+jd)}{\prod_{j=0}^{k-1}(b+w+jd)}$$

Question b) To obtain the distribution of S_n the number of black balls drawn, we note that there is $\binom{n}{k}$ different sequences each with the probability derived in question a) that lead to the event $S_n = k$.

$$P(S_n = k) = \binom{n}{k}\frac{\prod_{j=0}^{k-1}(b+jd)\prod_{j=0}^{n-k-1}(w+jd)}{\prod_{j=0}^{k-1}(b+w+jd)}$$

Question c)

$$\binom{n}{k}\frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}$$

Question d) Not independent since, but interchangeable

Question e) We approach the question by induction. We first show

$$P(X_1 = 1) = \frac{b}{b+w}$$

We then derive $P(X_{n+1} = 1)$ assuming $P(X_n = 1) = bb + w$ in a Polya model.

$$P(X_{n+1} = 1) = P(X_{n+1} = 1|X_1 = 1)P(X_1 = 1) + P(X_{n+1} = 1|X_1 = 0)P(X_1 = 0) = P(X_{n+1} = 1|X_1 =$$

To proceed we note that the probability $P(X_{n+1} = 1|X_1 = 1)$ is the probability of $P(Y_n = 1)$ in an urn scheme starting with $b+d$ blacks and w whites, thus $P(X_{n+1} = 1|X_1 = 1) = P(Y_n = 1) = \frac{b+d}{b+w+d}$. Correspondingly $P(X_{n+1} = 1|X_1 = 0) = \frac{b}{b+w+d}$. Finally

$$P(X_{n+1} = 1) = \frac{b+d}{b+w+d}\frac{b}{b+w} + \frac{b}{b+w+d}\frac{w}{b+w} = \frac{b}{b+w}$$

Question f)

$$P(X_5 = 1|X_{10} = 1) = \frac{P(X_{10} = 1|X_5 = 1)P(X_5 = 1)}{P(X_{10} = 1)} = P(X_{10} = 1|X_5 = 1)$$

using Bayes rule, or from the exchangeability. From the exchangeability we also have

$$P(X_{10} = 1|X_5 = 1) = P(X_2 = 1|X_1 = 1) = \frac{b + d}{b + w + d}$$

IMM - DTU

02405 Probability
2003-10-22
BFN/bfn

Solution for review exercise 34 (chapter 3) in Pitman

Question a) The function $g_z(x) = z^x$ defines a function of x for any $|z| < 1$. For fixed z we can find the $E(g_z(X))$ using the definition in the box on the top of page 175. We find

$$E(g_z(X)) = E(z^X) = \sum_{x=0}^{\infty} z^x P(X = x)$$

However, this is a power series in z that is absolutely convergent for $|z| \leq 1$ and thus defines a C^∞ function of z for $|z| < 1$.

Question b) The more elegant and maybe more abstract proof is

$$G_{X+Y}(z) = E\left(z^{X+Y}\right) = E\left(z^X z^Y\right)$$

From the independence of X and Y we get (page 177)

$$G_{X+Y}(z) == E\left(z^X\right) E\left(z^Y\right) = G_X(z) G_Y(z)$$

The more crude analytic proof goes as follows

$$G_{X+Y}(z) = E\left(z^{X+Y}\right) = \sum_{k=0}^{\infty} z^k P(X+Y = k) = \sum_{k=0}^{\infty} z^k \left(\sum_{i=0}^{k} P(X = i, Y = k-i)\right)$$

again from the independence of X and Y we get

$$G_{X+Y}(z) = \sum_{k=0}^{\infty} z^k \left(\sum_{i=0}^{k} P(X = i)P(Y = k-i)\right) \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} z^k P(X = i)P(Y = k-i)$$

The interchange of the sums are justified since all terms are positive. The rearrangement is a commonly used tool in analytic derivations in probability. It is quite instructive to draw a small diagram to verify the limits of the sums. We now make further rearrangements

$$G_{X+Y}(z) = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} z^k P(X = i)P(Y = k-i)$$

$$= \sum_{i=0}^{\infty} z^i P(X = i) \sum_{k=i}^{\infty} z^{k-i} P(Y = k-i) = \sum_{i=0}^{\infty} z^i P(X = i) \sum_{m=0}^{\infty} z^m P(Y = m)$$

by a change of variable ($m = k - i$). Now

$$G_{X+Y}(z) = \sum_{i=0}^{\infty} z^i P(X = i) \sum_{m=0}^{\infty} z^m P(Y = m) = \sum_{i=0}^{\infty} z^i P(X = i) G_Y(z) = G_X(z) G_Y(z)$$

Question c) By rearranging $S_n = (X_1 + \cdots + X_{n-1}) + X_n$ we deduce

$$G_{S_n}(z) = \prod_{i=1}^{n} G_{X_i}(z)$$

We first find the generating function of a Bernoulli distributed random variable(binomial with $n = 1$)

$$E(z^X) = \sum_{x=0}^{1} z^x P(X = x) = z^0 \cdot (1-p) + z^1 \cdot p = 1 - p(1-z)$$

Now using the general result for X_i with binomial distribution $b(n_i, p)$ we get

$$E(z^{X_i}) = (E(z^X))^{n_i} = (1 - p(1-z))^{n_i}$$

Generalizing this result we find

$$E(z^{S_n}) = (1 - p(1-z))^{\sum_{i=1}^{n} n_i}$$

i.e. that the sum of independent binomially distributed random variables is itself binomially distributed provided equality of the p_i's.

Question d) The generating function of the Poisson distribution is given in exercise 3.5.19. Such that

$$G_{S_n}(z) = \prod_{i=1}^{n} e^{-\mu_i(1-z)} = e^{-\sum_{i=1}^{n} \mu_i(1-z)}$$

The result proofs that the sum of independent Poisson random variables is itself Poisson.

Question e)

$$G_X(z) = \frac{zp}{1 - z(1-p)} \qquad G_{S_n} = \left(\frac{zp}{1 - z(1-p)}\right)^n$$

Question f)

$$G_{S_n} = \left(\frac{zp}{1 - z(1-p)}\right)^{\sum_{i=1}^{n} r_i}$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for review exercise 7 (chapter 4) in Pitman

Question a) We require $\int_{\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \alpha e^{-\beta|x|}\mathrm{d}x = 1$. We have $\alpha = \frac{\beta}{2}$ since $\int_0^{\infty} \beta e^{-\beta x}\mathrm{d}x = 1$.

Question b) We immediately get $E(X) = 0$ since $f(x)$ is symmetric around zero. The second moment $E(X^2)$ is identical to the second moment of the standard exponential, which we can find from the computational formula for the variance. We additionally have $Var(X) = E(X^2)$ since $E(X) = 0$.

$$Var(X) = E(X^2) = \frac{1}{\beta^2} + \left(\frac{1}{\beta}\right)^2 = \frac{2}{\beta^2}$$

Question c)

$$P(|X| > y) = 2P(X > y) = 2\int_y^{\infty} \frac{\beta}{2} e^{-\beta t}\mathrm{d}t = \int_y^{\infty} \beta e^{-\beta t}\mathrm{d}t = e^{-\beta y}$$

the standard exponential survival function.

Question d) From the result in c) we are lead to

$$P(X \leq x) = \begin{cases} \frac{1}{2}e^{\beta x} & x < 0 \\ 0.5 + \frac{1}{2}e^{-\beta x} & 0 < x \end{cases}$$

IMM - DTU

02405 Probability
2003-10-16
BFN/bfn

Solution for review exercise 13 (chapter 4) in Pitman

We introduce the random variables $N_{\text{loc}}(t)$ and $N_{\text{dis}}(t)$ as the number of local respectively long distance calls arriving within time t (where t is given in minutes).

Question a)

$$P(N_{\text{loc}}(1) = 5, N_{\text{dis}}(1) = 3) = P(N_{\text{loc}}(1) = 5)P(N_{\text{dis}}(1) = 3)$$

due to the independence of the Poisson processes. The variables $N_{\text{loc}}(t)$ and $N_{\text{dis}}(t)$ has Poisson distributions (page 289) such that

$$P(N_{\text{loc}}(1) = 5, N_{\text{dis}}(1) = 3) = \frac{(\lambda_{\text{loc}} \cdot 1)^5}{5!} e^{-\lambda_{\text{loc}} \cdot 1} \cdot \frac{(\lambda_{\text{dis}} \cdot 1)^3}{3!} e^{-\lambda_{\text{dis}} \cdot 1} = \frac{\lambda_{\text{dis}}^3 \lambda_{\text{loc}}^5}{5!3!} e^{-\lambda_{\text{loc}} - \lambda_{\text{dis}}}$$

Question b) The sum of two indpendent Poisson random variables is Poisson distributed (boxed result page 226), leading to

$$P(N_{\text{loc}}(3) + N_{\text{dis}}(3) = 50) = \frac{((\lambda_{\text{loc}} + \lambda_{\text{dis}})3)^{50}}{50!} e^{-(\lambda_{\text{loc}} + \lambda_{\text{dis}})3}$$

Question c) We now introduce the random variables Si_{loc} and Si_{dis} as the time of the i'th local and long distance call respectively. These random variables are Gamma distributed according to the box on the top of page 286 or to 4. page 289 The probability in question can be expressed as The waiting time to the first long distance in terms of calls are geometrically distributed

$$P(X > 10) = (1 - p_{\text{dis}})^{10} = \left(\frac{\lambda_{\text{loc}}}{\lambda_{\text{loc}} + \lambda_{\text{dis}}} \right)^{10}$$

IMM - DTU

02405 Probability
2003-11-2
BFN/bfn

Solution for review exercise 21 (chapter 4) in Pitman

Question a) We first note using exercise 4.3.4 page 301 and exercise 4.4.9 page 310 hat R_1 and R_2 are both Weibull$\left(\alpha = 2, \lambda = \frac{1}{2}\right)$ distributed. The survival function is thus (from E4.3.4) $G(x) = e^{-\frac{1}{2}x^2}$. We now apply the result for the minimum of independent random variables page 317 to get

$$P(Y \leq y) = P(\min(R_1, R_2) \leq y) = 1 - P(R_1 > y, R_2 > y) = 1 - P(R_1 > y)(R_2 > y)$$

$$= 1 - e^{-\frac{1}{2}y^2} e^{-\frac{1}{2}y^2} = 1 - e^{-y^2}$$

a new Weibull distribution with $\alpha = 2$ and $\lambda = 1$. If we did not recognize the distribution as a Weibull we would derive the survival function of the R_i's by

$$P(R_i > x) = \int_x^\infty u e^{-\frac{1}{2}u^2} \mathrm{d}u = e^{-\frac{1}{2}x^2}$$

We find the density using (5) page 297 or directly using E4.3.4 (i)

$$f_Y(y) = 2ye^{-y^2}$$

Question b) This is a special case of E4.4.9 a). We can re-derive this result using the change of variable formula page 304. With $Z = g(Y) = Y^2$ we get $\frac{\mathrm{d}g(y)}{\mathrm{d}y} = 2y$. Inserting we get

$$f_Z(z) = 2ye^{-y^2} \frac{1}{2y} = e^{-z}$$

an exponential(1) distribution.

Question c) We have $E(Z) = 1$ (see e.g. the mean of an exponential variable page 279 or the distribution summary page 477 or page 480).

IMM - DTU

02405 Probability
2003-10-16
BFN/bfn

Solution for review exercise 23 (chapter 4) in Pitman

We introduce $Y = M - 3$ such that Y has the exponential distribution with mean 2.

Question a)

$$E(M) = E(Y+3) = E(Y) + 3 = 5 \qquad Var(M) = Var(Y+3) = Var(Y) = 4$$

where we have used standard rules for mean and variance see eg. page 249, and the result page 279 for the variance of the exponential distribution.

Question b) We get the density $f_M(m)$ of the random variable M is

$$f_M(m) = \frac{1}{2}e^{-\frac{1}{2}(m-3)} \qquad m > 3.$$

from the stated assumptions. We can apply the box page 304 to get

$$f_X(x) = \frac{f_M(m)}{\frac{\mathrm{d}x}{\mathrm{d}m}} = \frac{\frac{1}{2}e^{-\frac{1}{2}(\log(x)-3)}}{x} = \frac{\frac{e^{\frac{3}{2}}}{2}}{x\sqrt{x}}, \qquad x > e^3$$

where $X = g(M) = e^M$. Alternatively

$$F_X(x) = P(X \le x) = P(\log(X) \le \log(x) = P(\log(X) - 3 \le \log(x) - 3)$$

$$= P(Y \le \log(x) - 3) = 1 - e^{\frac{-(\log(x)-3)}{2}} = 1 - \frac{e^{\frac{3}{2}}}{\sqrt{x}} \qquad x > e^3$$

taking derivative we get

$$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x} == \frac{\frac{e^{\frac{3}{2}}}{2}}{x\sqrt{x}}, \qquad x > e^3$$

Question c) We do the calculations in terms of the random variables $Y_i = M_i - 3$, $M_i = \log(X_i)$. Here X_i denotes the magnitude of the i'th earthquake. From Example 3 page 317 we know that the minimum Z of the Y_i's, $Z = \min(Y_1, Y_2)$ is exponentially distributed with mean 1.

$$P(M > 4) = P(Z > 1) = e^{-1}$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for review exercise 25 (chapter 4) in Pitman

Question a) We first note that the range of Y is $0 < Y \leq \frac{1}{2}$.

$$P(Y \leq y) = P\left(U \leq \frac{1}{2}\right) P\left((Y \leq y | U \leq \frac{1}{2}\right) + P\left(\frac{1}{2} < U\right) P\left((Y \leq y | \frac{1}{2} < U\right) = 2P(U \leq y)$$

The density is 2 for $0 < y < \frac{1}{2}$ 0 elsewhere.

Question b) The standard uniform density $f(y) = 1$ for $0 < y < 1$, 0 elsewhere.

Question c)

$$E(Y) = \frac{\frac{1}{2} - 0}{2} = \frac{1}{4}, Var(Y) = \frac{\left(\frac{1}{2} - 0\right)^2}{12} = \frac{1}{48}$$

IMM - DTU

02405 Probability
2003-11-12
BFN/bfn

Solution for review exercise 26 (chapter 4) in Pitman

Question a)

$$E(W_t) = E\left(Xe^{tY}\right) = E(X)E\left(e^{tY}\right)$$

by the independence of X and Y. We find $E\left(e^{tY}\right)$ from the definition of the mean.

$$E\left(e^{tY}\right) = \int_1^{\frac{3}{2}} e^{ty} \cdot 2\mathrm{d}y = \frac{2e^t}{t}\left(e^{\frac{t}{2}} - 1\right)$$

Inserting this result and $E(X) = 2$ we get

$$E(W_t) = 2\frac{2e^t}{t}\left(e^{\frac{t}{2}} - 1\right)$$

Alternatively we could derive the joint density of X and Y to

$$f(x, y) = 2(2x)^3 e^{-2x}, \qquad 0 < x, 0 < y < 1$$

where we have used that X has Gamma (4,2) density, and apply the formula for $E(g(X, Y))$ page 349.

Question b) Since X and Y are independent we find $E(W_t^2)$

$$E(W_t^2) = E(X^2)E\left(\left(e^{tY}\right)^2\right)$$

where $E(X^2) = Var(X) + (E(X))^2 = 5$, see eg. page 481. Next we derive

$$E\left(\left(e^{tY}\right)^2\right) = \frac{e^{2t}}{t}\left(e^t - 1\right)$$

and apply the computational formula for the variance page 261

$$SD(W_t) = \sqrt{5\frac{e^{2t}}{t}\left(e^t - 1\right) - \left(2\frac{2e^t}{t}\left(e^{\frac{t}{2}} - 1\right)\right)^2} =$$

IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for review exercise 1 (chapter 5) in Pitman

First apply the definition of conditional probability page 36

$$P\left(Y \geq \frac{1}{2} | Y \geq X^2\right) = \frac{P\left(Y \geq \frac{1}{2} \cap Y \geq X^2\right)}{P(Y \geq X^2)}$$

The joint density of X and Y is the product of the marginal densities since X and Y are independent (page 349). We calculate the denominator using the formula for the probability of a set B page 349

$$P(Y \geq X^2) = \int_0^1 \int_{x^2}^1 1 \cdot 1 \cdot \mathrm{d}y\mathrm{d}x = \int_0^1 (1 - x^2)\mathrm{d}x = 1 - \frac{1}{3} = \frac{2}{3}$$

and the numerator

$$P\left(Y \geq \frac{1}{2} \cap Y \geq X^2\right) = P(Y \geq X^2) - P\left(Y < \frac{1}{2} \cap Y \geq X^2\right)$$

Now for the last term

$$P\left(Y < \frac{1}{2} \cap Y \geq X^2\right) = \int_0^{\frac{1}{\sqrt{2}}} \int_{x^2}^{\frac{1}{2}} 1 \cdot \mathrm{d}y\mathrm{d}x = \int_0^{\frac{1}{\sqrt{2}}} (\frac{1}{2} - x^2)\mathrm{d}x$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{1}{3}\frac{1}{2}\frac{1}{\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

Finally we get

$$P\left(Y \geq \frac{1}{2} | Y \geq X^2\right) = \frac{\frac{2}{3} - \frac{1}{3\sqrt{2}}}{\frac{2}{3}} = 1 - \frac{\sqrt{2}}{4}$$

IMM - DTU

02405 Probability
2003-10-17
BFN/bfn

Solution for review exercise 20 (chapter 5) in Pitman

Question a) This is example 3 page 317. A rederivation gives us

$$P(T_{\min} \leq t) = 1 - P(T_{\min} > t) = 1 - P(T_1 > t, T_2 > t)$$

with T_1 and T_2 independent we get

$$P(T_{\min} \leq t) = 1 - P(T_1 > t)P(T_2 > t)$$

now inserting the exponential survival function page 279 we get

$$P(T_{\min} \leq t) = 1 - \left(1 - \left(1 - e^{-\lambda_1 t}\right)\right)\left(1 - \left(1 - e^{-\lambda_2 t}\right)\right) = 1 - e^{-(\lambda_1+\lambda_2)t}$$

the cumulative distribution function of an exponentially distributed random variable with parameter $\lambda_1 + \lambda_2$.

Question b) This question is Example 2 page 352. A slightly different handling of the integrals gives us

$$P(T_1 < T_2) = \int_0^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \mathrm{d}t_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \int_0^\infty f_{T_1}(t_1) P(T_2 > t_1) \mathrm{d}t_1$$

which is an application of the rule of averaged conditional probability (page 41) for a continuous density. The general result is stated page 417 as the Integral Conditioning Formula. We get

$$P(T_1 < T_2) = \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Question c) Consider

$$P(T_{\min} > t | X_{\min} = 2) = P(T_1 > t | T_2 > T_1) = \frac{P(T_1 > t, T_2 > T_1)}{P(T_2 > T_1)} = \frac{P(T_1 > t, T_2 > T_1)}{P(X_{\min} = 2)}$$

We evaluate the probability in the denominator by integrating the joint density over a proper region (page 349), similarly to example 2 page 352

$$P(T_1 > t, T_2 > T_1) = \int_t^\infty \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \mathrm{d}t_2 \mathrm{d}t_1$$

$$= \int_t^{\infty} \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \mathrm{d}t_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

By inserting back we finally get

$$P(T_{\text{min}} > t | X_{\text{min}} = 2) = e^{-(\lambda_1+\lambda_2)t} = P(T_{\text{min}} > t)$$

such that T_{min} and X_{min} are independent.

Question d) We can define $X_{\text{min}} = i$ whenever $T_{\text{min}} = T_i$. Then $P(X_{\text{min}} = i) = \frac{\lambda_i}{\lambda_1+\cdots+\lambda_n}$, and T_{min} and X_{min} are independent.

IMM - DTU

02405 Probability
2004-5-13
BFN/bfn

Solution for review exercise 8 (chapter 6) in Pitman

Question a) Using the multiplication rule see e.g. page 425 top we get

$$f(x,y) = f_Y(y)f_X(x|Y=y) = 2e^{-2y}\frac{1}{y}e^{-\frac{x}{y}}$$

The marginal density $f_X(x)$ of X is given by

$$f_X(x) = \int_0^\infty 2e^{-2y}\frac{1}{y}e^{-\frac{x}{y}}\mathrm{d}y$$

a non-standard density.

Question b) Using average conditional expectation page 425 bottom we get

$$E(X) = E(E(X|Y)) = E(Y) = \frac{1}{2}$$

noting that the roles of X and Y are interchanged.

Question c) Similarly

$$E(XY) = E(E(XY|Y)) = E(YE(X|Y)) = E(Y^2) = Var(Y) + (E(Y))^2 = \frac{1}{2}$$

We have $E(X^2) = E(E(X^2|Y)) = E(2Y^2) = 1$. Thus $Var(X) = SD(X)^2 = 1 - \frac{1}{4} = \frac{3}{4}$ and $SD(Y) = \frac{1}{2}$. Finally $Corr(X,Y) = \frac{\frac{1}{2}-\frac{1}{4}}{\frac{\sqrt{3}}{2}\frac{1}{2}} = \frac{\sqrt{3}}{3}$