IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 6.5.10 in Pitman

Question a) We first note from page that since V are W are bivariate normal, then

$$X = \frac{V - \mu_V}{\sigma_V} \qquad Y = \frac{W - \mu_W}{\sigma_W}$$

are bivariate standardized normal. From page we have that we can write

$$Y = \rho X + \sqrt{1-\rho^2} Z$$

where X and Z are standardized independent normal variables. Thus any linear combination of V and W will be a linear combination of X and Z. We know from chapter 5. that such a combination is a normal variable. After some tedious calculations we find the actual linear combinations to be

$$aV + bW = a\mu_V + b\mu_W + (a\sigma_V + b\rho\sigma_W)X + b\sigma_2\sqrt{1-\rho^2}Z$$

and

$$cV + dW = c\mu_V + d\mu_W + (c\sigma_V + d\rho\sigma_W)X + d\sigma_2\sqrt{1-\rho^2}Z$$

Such that $(aV + bW \in normal(a\mu_V + b\mu_W, a^2\sigma_V^2 + b^2\sigma_W^2 + 2ab\rho\sigma_V\sigma_W)$ and $(cV + dW \in normal(c\mu_V + d\mu_W, c^2\sigma_V^2 + d^2\sigma_W^2 + 2cd\rho\sigma_V\sigma_W)$.

Question b) We have from question a) that

$$V_1 = aV + bW = \mu_1 + \gamma_{11}X + \gamma_{12}Z \qquad W_1 = cV + dW = \mu_2 + \gamma_{21}X + \gamma_{22}Z$$

for some appropriate constants. We can rewrite these expressions to get

$$\frac{V_1 - \mu_1}{\sqrt{\gamma_{11}^2 + \gamma_{12}^2}} = \frac{\gamma_{11}X + \gamma_{12}Z}{\sqrt{\gamma_{11}^2 + \gamma_{12}^2}} = X_1 \qquad \frac{W_1 - \mu_2}{\sqrt{\gamma_{21}^2 + \gamma_{22}^2}} = \frac{\gamma_{21}X + \gamma_{22}Z}{\sqrt{\gamma_{21}^2 + \gamma_{22}^2}} = Y_1$$

such that X_1 and Y_1 are standard normal variables. We see that with some effort we would be able to write

$$Y_1 = \rho_1 X_1 + \sqrt{1-\rho_1^2} Z_1$$

and we conclude from page 454 that V_1 and W_2 are bivariate normal variables.

Question c) We find the parameters using standard results for mean and variance

$$\mu_1 = E(aV + bW) = a\mu_V + b\mu_W \qquad \mu_2 = E(cV + dW) = c\mu_V + b\mu_W$$

$$\sigma_1^2 = a^2\sigma_V^2 + b^2\sigma_W^2 + 2ab\rho\sigma_V\sigma_W \qquad \sigma_2^2 = c^2\sigma_V^2 + d^2\sigma_W^2 + 2cd\rho\sigma_V\sigma_W$$

We find the covariance from

$$E((aV + bW - (a\mu_V + b\mu_W))(cV + dW - (c\mu_V + d\mu_W)))$$

$$= E[(a(V - \mu_V) + b(W - b\mu_W))(c(V - \mu_V) + d(W - \mu_W))]$$

etc