IMM - DTU

02405 Probability
2003-11-19
BFN/bfn

Solution for exercise 6.3.14 in Pitman

We have immediately

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i}$$

The posterior density of p given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{f(p; X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}$$

$$= \frac{f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|p)f(p)}{\int_0^1 f(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|p)f(p)\mathrm{d}p}$$

Inserting the previous result to get

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i} f(p)}{\int_0^1 p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i} f(p)\mathrm{d}p}$$

which only dependes on the X_i's through their sum. Introducing $S_n = \sum_{i=1}^n X_i$ we rewrite

$$f(p|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{p^{S_n}(1-p)^{n-S_n} f(p)}{\int_0^1 p^{S_n}(1-p)^{n-S_n} f(p)\mathrm{d}p}$$

We note that if the prior density of p $f(p)$ is a $beta(r, s)$ distribution, then the posterior distribution is a $beta(r + S_n, s + n - S_n)$ distribution.