IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 6.2.5 in Pitman

Question a) By the defition of the c.d.f. page 311

$$F(x) = P(X \le x) = P(X \le x|A)P(A) + P(X \le x|A^c)P(A^c)$$

using the rule of averaged conditional probabilities page 41. Now introduce the parameters of the exercise ($P(X \le x|A) = F_1(x)$ etc.) to get

$$F(x) = p \cdot F_1(x) + (1-p)F_2(x)$$

Question b) We first find the density of X assuming X_1 and X_2 continuous (similar calculations can be made in full generality)

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = p \cdot f_1(x) + (1-p) \cdot f_2(x)$$

with $f_i(x) = \frac{\mathrm{d}F_i(x)}{\mathrm{d}x}, i = 1, 2$. Now (see e.g. page 261 top)

$$E(X) = \int x \cdot f(x)\mathrm{d}x = \int x(p \cdot f_1(x) + (1-p) \cdot f_2(x))\mathrm{d}x$$

using the linearity of the integral we get

$$= p \int x \cdot f_1(x)\mathrm{d}x + (1-p) \int x \cdot f_2(x)\mathrm{d}x = p \cdot E(X_1) + (1-p) \cdot E(X_2)$$

Question c) We first note that we can derive $E(X^2)$ in a similar way, thus

$$E(X^2) = p \cdot E(X_1^2) + (1-p) \cdot E(X_2^2) = p \cdot (Var(X_1) + E(X_1)^2) + (1-p) \cdot (Var(X_2) + E(X_2)^2)$$

where we have used the computational formula for the variance e.g. page 261. Applying this formula once more we get

$$Var(X) = E(X^2) - E(X)^2$$

$$= p \cdot (Var(X_1) + E(X_1)^2) + (1-p) \cdot (Var(X_2) + E(X_2)^2) - (p \cdot E(X_1) + (1-p) \cdot E(X_2))^2$$

$$= p \cdot Var(X_1) + (1-p) \cdot Var(X_2) + p(1-p)(E(X_1) - E(X_2))^2$$