IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 6.2.18 in Pitman

By definition

$$Var(Y) = \sum_y (y - E(Y))^2 f(y) = \sum_y (y - E(Y))^2 \sum_x f(x,y) = \sum_x \sum_y (y - E(Y))^2 f(x,y)$$

We now apply the crucial idea of adding 0 in the form of $E(Y|x) - E(Y|x)$ inside the brackets.

$$Var(Y) = \sum_x \sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 f(x,y)$$

Next we multiply with one in the form of $\frac{f(x)}{f(x)}$

$$Var(Y) = \sum_x \sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 \frac{f(x,y)}{f(x)} f(x)$$

By definition $f_Y(y|x) = \frac{f(x,y)}{f(x)}$ thus

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x) + E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

Expanding the square sum we get

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x))^2 + (E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

since $\sum_y (y - E(Y|x)) = 0$. Now

$$Var(Y) = \sum_x \left[\sum_y (y - E(Y|x))^2 f_Y(y|x) \right] f(x) + \sum_x \left[\sum_y (E(Y|x) - E(Y))^2 f_Y(y|x) \right] f(x)$$

the inner part of the first term is $Var(Y|X = x)$ while the inner part of the second term is constant. Thus

$$Var(Y) = \sum_x Var(Y|X = x) f(x) + \sum_x (E(Y|x) - E(Y))^2 f(x)$$

leading to the stated equation

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

an important and very useful result that is also valid for continuous and mixed distributions. Mixed distributions are distributions that are neither discrete nor continuous.