IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 6.1.6 in Pitman

Question a) We recall the definition of conditional probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$, such that

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) \frac{P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m \cap \sum_{i=1}^{m} N_i = n)}{P(\sum_{i=1}^{m} N_i = n)}$$

Now realising that $P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m \cap \sum_{i=1}^{m} N_i = n) = P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m)$ and using the fact that $N = \sum_{i=1}^{m} N_i$ has Poisson distribution with parameter $\lambda = \sum_{i=1}^{m} \lambda_i$ we get

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) = \frac{\prod_{i=1}^{m} \frac{\lambda_i^{n_i}}{n_i!} e^{-\lambda_i}}{\frac{\lambda^{\sum_{i=1}^{m} n_i}}{(\sum_{i=1}^{m} n_i)!} e^{-\lambda}}$$

such that with $n = \sum_{i=1}^{m} n_i$

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n) = \frac{n!}{n_1! n_2! \cdots n_m!} \left(\frac{\lambda_1}{\lambda}\right)^{n_1} \left(\frac{\lambda_1}{\lambda}\right)^{n_2} \cdots \left(\frac{\lambda_m}{\lambda}\right)^{n_m}$$

a multinomial distribution (page 155) with probabilities $p_i = \frac{\lambda_i}{\lambda}$.

Question b) Using

$$P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m) = P(N = n) P(N_1 = n_1, N_2 = n_2, \ldots N_m = n_m | \sum_{i=1}^{m} N_i = n)$$

we see that the N_i's are independent Poisson variables.