Solution for exercise 6.1.1 in Pitman

Question a) X is binomially distributed $b\left(3, \frac{1}{2}\right)$.

$$P(X=0)=\frac{1}{8}, P(X=1)=\frac{3}{8}, P(X=2)=\frac{3}{8}, P(X=3)=\frac{1}{8}$$

Question b) We introduce the random variables Z_x with binomial distribution $b\left(3-x, \frac{1}{2}\right)$. We can write $Y = x+Z_x$ for the conditional distribution of Y. For $x=0$ we get

$$P(Y=0|X=0)=\frac{1}{8}, P(Y=1|X=0)=\frac{3}{8}, P(Y=2|X=0)=\frac{3}{8}, P(Y=3|X=0)=\frac{1}{8}$$

For $x=1$ we get

$$P(Y=1|X=1)=\frac{1}{4}, P(Y=2|X=1)=\frac{1}{2}, P(Y=3|X=1)=\frac{1}{4}$$

For $x=2$ we get

$$P(Y=2|X=2)=\frac{1}{2}, P(Y=3|X=2)=\frac{1}{2}$$

For $x=3$ we get

$$P(Y=3|X=3)=1$$

Question c) We find $P(X=x, Y=y) = P(X=x)P(Y=y|X=x)$. The distribution table is

X/Y	0	1	2	3
0	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$
1	0	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{3}{32}$
2	0	0	$\frac{3}{16}$	$\frac{3}{16}$
3	0	0	0	$\frac{1}{8}$

Question d) We find the distribution of Y from the distribution table in the previous question

$$P(Y=0)=\frac{1}{64}, P(Y=1)=\frac{9}{64}, P(Y=2)=\frac{27}{64}, P(Y=3)=\frac{27}{64}$$

Question e) Using $P(X=x|Y=y)=\frac{P(X=x,Y=y)}{P(Y=y)}$ we get for $y=0$

$$P(X=0|Y=0)=1$$

for $y = 1$

$$P(X = 0|Y = 1) = \frac{1}{3}, P(X = 1|Y = 1) = \frac{2}{3}$$

for $y = 2$

$$P(X = 0|Y = 2) = \frac{1}{9}, P(X = 1|Y = 2) = \frac{4}{9}, P(X = 2|Y = 2) = \frac{4}{9}$$

for $y = 3$

$$P(X = 0|Y = 3) = \frac{1}{27}, P(X = 1|Y = 3) = \frac{2}{9}, P(X = 2|Y = 3) = \frac{4}{9}, P(X = 3|Y = 3) = \frac{8}{27}$$

Question f) Best guess $\hat{X}_y$ of $X|Y = y$

$Y = y$	0	1	2	3
$\hat{X}_y$	0	1	1 or 2	2

Question g)

$$\sum_{y=0}^{3} P(Y = y)P(X = \hat{X}_y) = \frac{1}{64} \cdot 1 + \frac{9}{64} \cdot \frac{2}{3} + \frac{27}{64} \cdot \frac{4}{9} + \frac{27}{64} \cdot \frac{4}{9} = \frac{31}{64}$$