IMM - DTU

02405 Probability
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BFN/bfn

Solution for exercise 5.4.19 in Pitman

We apply the technique of the proof for the distribution of ratios formula page 382-383. Define $Z = \frac{X}{X+Y}$. The event $z < Z < z + \mathrm{d}z$ occurs whenever Y is between the two lines $\frac{x}{x+y} = z + \mathrm{d}z$ and $\frac{x}{x+y} = z$. We get the length of the vertical side of the rectangle by solving for y in the two equations above. Thus

$$y_2 = \left(\frac{1}{z} - 1\right)x, y_1 = \left(\frac{1}{z + \mathrm{d}z} - 1\right)x, \qquad y_2 - y_1 = \frac{x\mathrm{d}z}{z(z + \mathrm{d}z)} = \tilde{=} \frac{x\mathrm{d}z}{z^2}$$

We have derived a general formula for the density of $Z = \frac{X}{X+Y}$ for non negative X and Y

$$\int_0^\infty \frac{x}{z^2} f_X(x) f_Y\left(\frac{(1-z)x}{z}\right)\mathrm{d}x$$

We now insert the gamma densities of X and Y (see page 481) to get

$$\int_0^\infty \frac{x}{z^2}\lambda\frac{(\lambda x)^{r-1}}{\Gamma(r)}e^{-\lambda x}\lambda\frac{\left(\lambda\frac{(1-z)x}{z}\right)^{s-1}}{\Gamma(s)}e^{-\lambda\frac{(1-z)x}{z}}\mathrm{d}x$$

We simplify to get

$$\frac{1}{z^2\Gamma(r)\Gamma(s)}\left(\frac{1-z}{z}\right)^{s-1}\int_0^\infty \lambda(\lambda x)^{r+s-1}e^{-\lambda\frac{x}{z}}\mathrm{d}x$$

the function under the integral is very close to a gamma density such that with

$$\frac{1}{z^2\Gamma(r)\Gamma(s)}\left(\frac{1-z}{z}\right)^{s-1}\Gamma(r+s)z^{r+s-1}\int_0^\infty \lambda\frac{\left(\frac{(\lambda x)}{z}\right)^{r+s-1}}{\Gamma(r+s)}e^{-\lambda\frac{x}{z}}\mathrm{d}x$$

we get the density of a *gamma* $\left(r+s, \frac{\lambda}{z}\right)$ variable. Thus

$$f_Z(z) = \frac{\Gamma(r+s}{\Gamma(r)\Gamma(s)}(1-z)^{s-1}z^{r+s-(s-1)-2} = \frac{1}{B(r,s)}z^{r-1}(1-z)^{s-1}$$

the density of a *beta*(r,s) random variable.

Independence

Three lines to follow

1. We see directly from the calculations

2. Considering

$$P\left(z < \frac{X}{X+Y} < z + \mathrm{d}z | w < X + Y < w + \mathrm{d}w\right) = \frac{P\left(z < \frac{X}{X+Y} < z + \mathrm{d}z, w < X + Y < w + \mathrm{d}w\right)}{P(w < X + Y < w + \mathrm{d}w)}$$

3. Using the division rule page 425