IMM - DTU

02405 Probability
2003-11-1
BFN/bfn

Solution for exercise 5.2.15 in Pitman

Question a)

$$P(a < X \leq b, c < Y \leq d) = P(X \leq b, c < Y \leq d) - P(X \leq a, c < Y \leq d)$$

$$= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - (P(X \leq a, Y \leq d) - P(X \leq a, Y \leq c))$$
$$= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - P(X \leq a, Y \leq d) + P(X \leq a, Y \leq c)$$
$$= F(b, d) - F(b, c) - F(a, d) + F(a, c)$$

This relation can also be derived from geometric considerations.

Question b)

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) \mathrm{d}u \mathrm{d}v$$

Question c)

$$f(x, y) = \frac{d^2 F(x, y)}{\mathrm{d}x \mathrm{d}y}$$

from the fundamental theorem of calculus.

Question d) The result follows from (2) page 350 by integration.

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(x) f_Y(y) \mathrm{d}y \mathrm{d}x = \int_{-\infty}^{x} f_X(x) \mathrm{d}x \int_{-\infty}^{y} f_Y(y) \mathrm{d}y = F_X(x) F_Y(y)$$

Alternatively define the indicator $I(x, y)$ variables such that $I(x, y) = 1$ if $X \leq x$ and $Y \leq y$ and 0 otherwise. Note that $F(x, y) = P(I(x, y) = 1) = E(I(x, y))$ and apply the last formula on page 349.

Question e) See also exercise 4.6.3 c). We find

$$F(x, y) = P(U_{(1)} \leq x, U_{(n)} \leq y) = P(U_{(n)} \leq y) - P(U_{(1)} > x, U_{(n)} \leq y)$$

$$P(U_{(n)} \leq y) - P(x < U_1 \leq y, x < U_2 \leq y, \ldots, x < U_n \leq y) = y^n - (y - x)^n$$

We find the density as

$$\frac{d^2 F(x, y)}{\mathrm{d}x \mathrm{d}y} = n(n-1)(y - x)^{n-2}$$